# A periodic billiard path in every polygon

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A new preprint by Giovanni Forni (Forni 2026) proves a beautiful theorem which is very easy to state:

**Theorem 1** *Every finite bounded polygon has a regular periodic billiard orbit.*

This means the following. Put an ideal billiard ball inside any polygonal table. The ball moves in a straight line, and when it hits a side it bounces in the usual way: angle in equals angle out. The theorem says that there is always at least one path which eventually repeats exactly. The word regular only means that the path never hits a corner. This condition is necessary, because if the ball hits a corner, the reflection rule is no longer well-defined.

This problem is surprisingly hard. For some special polygons the answer was known before. For example, acute triangles have a classical periodic path, and polygons with rational angles can be studied using the theory of flat surfaces. Forni's theorem has no such restriction: the angles of the polygon can be arbitrary.

For the proof, Forni first uses the standard trick of unfolding the billiard table. Instead of making the ball reflect from a side, reflect the whole polygon across that side; then the broken billiard path becomes one straight line. After doing this repeatedly, the problem becomes straight-line motion on a flat surface made from copies of the polygon. The only special places on this surface come from the corners of the polygon: near such a point the surface looks like the tip of a cone, or like several paper sectors glued around one point. In Forni's terminology these special points are conical singularities.

Forni then argues by contradiction. If there were no repeating path, then a theorem of Galperin, Kruger, and Troubetzkoy (Galperin et al. 1995) would force every straight path to pass arbitrarily close to a cone tip. Forni turns this into a statement about the three-dimensional space of all positions and directions: this space would collapse toward the boundary pieces coming from the cone tips. He then studies the skeleton where different boundary pieces are equally close. The no-periodic-path assumption would force this skeleton to have more and more independent loops, but a general topological theorem

(Vaĭnšteĭn et al. 1978) says that the skeleton cannot have more loop-complexity than the fixed original space. This contradiction proves that a periodic billiard path must exist.

What is striking is that the proof does not construct the periodic path. It proves that a universe with no periodic path would have impossible topology: the hidden skeleton of the phase space would need to have both bounded and unbounded loop-complexity at the same time.

# References

Forni, Giovanni. 2026. “Existence of a Periodic Orbit for Billiards in Polygons.” *arXiv Preprint arXiv:2606.10102*.

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