On De Giorgi’s conjecture in dimensions up to 8

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see this post.

My choice for 2003 in Analysis is Ovidiu Savin’s proof of a major case of the famous conjecture of De Giorgi for the Allen–Cahn equation. More precisely, Savin proved the conjecture in dimensions up to 8 under an additional limiting assumption. The paper was received by the *Annals of Mathematics* in 2003 and published in 2009 (Savin 2009).

Many of the most beautiful problems in analysis come from equations that are simple to state but remarkably hard to understand. Often such equations arise in physics, where they describe a phenomenon whose qualitative behavior is clear at an intuitive level, while the corresponding mathematics remains mysterious for decades. A particularly important example is the *Allen–Cahn equation*

$$\Delta u = u^3 - u, \tag{1}$$

where $u : \mathbb{R}^n \to \mathbb{R}$ and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian. This equation is one of the standard models for phase transitions. One should imagine a medium which can be in two stable phases, represented by the values $u = -1$ and $u = 1$, with a thin transition layer between them. The equation says that the function u balances the tendency to vary smoothly, coming from the Laplacian, with the tendency to sit near one of the two preferred states.

The simplest nonconstant solution is the one-dimensional transition $q(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$, which satisfies $q'' = q^3 - q$, $\lim_{t\to-\infty} q(t) = -1$, $\lim_{t\to+\infty} q(t) = 1$.

Consequently, for any unit vector $e \in \mathbb{R}^n$ and any constant $c \in \mathbb{R}$, the function $u(x) = q(e \cdot x + c)$ solves (1). Its level sets are parallel hyperplanes. These are the planar transition layers. De Giorgi’s conjecture asks whether, under a natural monotonicity assumption, these obvious examples are the only ones in low dimensions.

In 1978, De Giorgi (De Giorgi 1978) conjectured that if $u : \mathbb{R}^n \to \mathbb{R}$ is a solution to (1) satisfying

1. $|u| < 1$, and
2. $\dfrac{\partial u}{\partial x_n} > 0$ for every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

then all level sets of u must be hyperplanes, at least when $n \leq 8$.

The monotonicity assumption says that, as one moves in the x_n-direction, the solution passes through the transition in a consistent direction. There are no folds or reversals. From a physical point of view, this is the assumption that the medium changes from one phase to the other as we move upward. From a geometric point of view, it says that the level sets of u are ordered graphs over the first $n-1$ variables. De Giorgi's conjecture says that, in low dimensions, such an ordered transition layer cannot bend: it must be flat.

The number 8 is not accidental. The conjecture is deeply connected with the theory of minimal surfaces and, in particular, with the Bernstein problem. Under a suitable rescaling, transition layers of Allen–Cahn solutions resemble minimal hypersurfaces. In the minimal surface world, dimension 8 is the critical threshold at which new phenomena begin to appear. De Giorgi's conjecture can therefore be viewed as an Allen–Cahn analogue of the rigidity of minimal graphs in low dimensions.

Before Savin's work, the conjecture had been proved only in small dimensions. In 1998, Ghoussoub and Gui (Ghoussoub and Gui 1998) proved it for $n = 2$. In 2000, Ambrosio and Cabré (Ambrosio and Cabré 2000) proved it for $n = 3$. These results already required substantial ideas from elliptic PDE, geometric analysis, and Liouville-type theorems. The remaining dimensions $4 \leq n \leq 8$ were much harder, precisely because the analogy with minimal surfaces becomes more subtle there.

Savin's theorem settled this range under the additional assumption that the solution actually connects the two pure phases at infinity in the monotone direction:

1. $\lim_{x_n \to \pm\infty} u(x_1, \dots, x_{n-1}, x_n) = \pm 1$ for every fixed $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

This hypothesis is very natural: it says that each vertical line sees a complete transition from the phase -1 to the phase 1. It rules out solutions that are monotone but do not genuinely connect the two wells of the potential.

Theorem 1 (Savin) *Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is a solution of the Allen–Cahn equation $\Delta u = u^3 - u$ satisfying* (i), (ii), *and* (iii) *above. If $n \leq 8$, then all level sets of u are hyperplanes.*

Equivalently, after a rotation and a translation, the solution is the one-dimensional profile $u(x) = \tanh\left(\frac{x_n}{\sqrt{2}}\right)$. Thus the complicated-looking partial differential equation (1), together with the monotonicity and limiting assumptions, admits no genuinely multidimensional transition layers in dimensions $n \leq 8$.

What makes the theorem so striking is the way it translates geometric intuition into a rigid analytic conclusion. The Allen–Cahn equation is an equation for a function, but Savin's theorem tells us that, under the right hypotheses, the geometry of its level sets is as rigid as the geometry of minimal hyperplanes. The proof does not merely solve an isolated PDE problem; it strengthens the bridge between phase transitions, variational methods, elliptic regularity, and minimal surface theory.

For me, Savin's result is one of the landmark achievements of early twenty-first century analysis. It takes a conjecture motivated by physics, reformulates it through the geometry of interfaces, imports the deep intuition of minimal surface theory, and produces a clean rigidity theorem: in low dimensions, a monotone phase transition that connects -1 to 1 must be flat.

References

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