A proof of the Stanley–Wilf conjecture

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see this post.

My choice for 2003 in combinatorics is the proof by Marcus and Tardos of the famous Stanley–Wilf conjecture, which states that the growth rate of every proper permutation class is singly exponential. The paper was submitted in 2003 and published in 2004 (Marcus and Tardos 2004).

A permutation of $[n] = 1, 2, \dots, n$ is called an n-*permutation*. We say that an n-permutation σ *contains* a k-permutation π if there exist integers $1 \leq x_1 < \dots < x_k \leq n$ such that for all $1 \leq i, j \leq k$ we have $\sigma(x_i) < \sigma(x_j)$ if and only if $\pi(i) < \pi(j)$. Otherwise, we say that σ *avoids* π. For a fixed permutation π, let $S_n(\pi)$ be the number of n-permutations avoiding π. In the late 1980s, Stanley and Wilf (Wilf 1993) conjectured that for all π there exists a constant c_π such that $S_n(\pi) \leq c_\pi^n$ for all n.

In 2000, Klazar (Klazar 2000) proved that this conjecture would follow from a conjecture of F''uredi and Hajnal about permutation matrices. Let A and P be 0-1 *matrices* (that is, matrices with all entries 0 or 1). We say that A *contains* the $k \times l$ matrix P with entries p_{ij} if there exists a $k \times l$ submatrix D of A with entries d_{ij} such that $d_{ij} = 1$ whenever $p_{ij} = 1$. Otherwise we say that A *avoids* P. Let $f(n, P)$ be the maximum number of 1-entries in an $n \times n$ 0-1 matrix avoiding P. A 0-1 matrix P is called a *permutation matrix* if it has exactly one entry of 1 in each row and each column and 0s elsewhere. In 1992, F''uredi and Hajnal (Füredi and Hajnal 1992) conjectured that $f(n, P)$ grows at most linearly in n. In 2004, Marcus and Tardos (Marcus and Tardos 2004) proved this conjecture.

Theorem 1 *For all permutation matrices P we have $f(n, P) = O(n)$.*

The proof of Theorem 1 is surprisingly simple. The authors used a partitioning of the larger matrix into blocks to prove a linear recursion for $f(n, P)$, from which the theorem follows easily. Together with the result of Klazar (Klazar 2000), Theorem 1 implies the truth of the Stanley–Wilf conjecture.

Theorem 1 became a cornerstone for many subsequent developments, one of which we describe below. In 2006, Balogh, Bollob'as and Morris (Balogh et al. 2006) formulated a far-reaching generalization of the Stanley–Wilf conjecture. An ordered graph is a graph (without loops or multiple edges) together with a total order $<$ on its vertices. A collection $\mathcal{P}$ of ordered graphs is called a property if it is closed under order-preserving isomorphisms of the vertex set. We will write $|\mathcal{P}_n|$ for the number of non-isomorphic ordered graphs on n vertices in $\mathcal{P}$. A property $\mathcal{P}$ of ordered graphs is called hereditary if it is closed under taking induced sub-graphs (with the induced ordering). Balogh, Bollob'as and Morris (Balogh et al. 2006) conjectured that for every hereditary property $\mathcal{P}$ of ordered graphs, the function $n \to |\mathcal{P}_n|$ either has at most exponential growth, or has at least factorial growth. In 2024, Bonnet et al. (Bonnet et al. 2024) confirmed this conjecture.

Theorem 2 *Let $\mathcal{P}$ be a hereditary property of ordered graphs. Then either $|\mathcal{P}_n| \leq c^n$ for some constant $c > 0$, or $|\mathcal{P} * n| \geq \sum *0 \leq k \leq n/2\binom{n}{2k}k! \geq \lfloor \frac{n}{2} \rfloor !$.*

The ordered permutation graph associated to a permutation π of $1, \ldots, n$ is the ordered graph G_π with vertices ordered as $1 < 2 < \cdots < n$, such that two vertices $i < j$ are adjacent if and only if $\pi(i) > \pi(j)$. The Stanley–Wilf conjecture corresponds to the special case of Theorem 2 when all graphs in $\mathcal{P}$ are ordered permutation graphs.

References

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