On Talagrand's convexity problem

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In a recent preprint (Hua et al. 2026), Hua, Song, and Tudose gave an affirmative answer to a famous question of Talagrand: can one create convexity from a large set in Gaussian space after only a bounded number of Minkowski additions?

For sets $A, B \subseteq \mathbb{R}^d$, write

$$A + B := \{a + b : a \in A,\ b \in B\}.$$

For a positive integer n, write

$$nA := \underbrace{A + \cdots + A}_{n \text{ times}}.$$

Thus nA is not the dilation of A, but the set of all sums of n points of A. If A is balanced, meaning that

$$x \in A, \quad |t| \leq 1 \qquad \Longrightarrow \qquad tx \in A,$$

then A is symmetric and contains the line segment from 0 to each of its points. By Carathéodory's theorem, every point of $\mathrm{conv}(A)$ is a convex combination of at most $d + 1$ points of A. Since A is balanced, the pieces of this convex combination already lie in A, and therefore

$$\mathrm{conv}(A) \subseteq (d + 1)A.$$

This gives a completely dimension-dependent way to create convexity. Talagrand's question asks whether a much weaker, but dimension-free, conclusion is true: instead of asking for the whole convex hull, can one at least find a large convex set inside nA, where n is a universal constant independent of d?

The largeness condition is measured using the standard Gaussian measure

$$\gamma_d(A) := \frac{1}{(2\pi)^{d/2}} \int_A e^{-|x|^2/2}\, dx, \qquad A \subseteq \mathbb{R}^d.$$

Talagrand asked whether there exists a positive integer n such that, for every dimension $d \geq 1$ and every balanced set $A \subseteq \mathbb{R}^d$ with

$$\gamma_d(A) \geq \frac{3}{4},$$

the set nA contains a convex set C satisfying

$$\gamma_d(C) \geq \frac{1}{2}.$$

The point is that n should be universal: it should not depend on d, on A, or on any geometric complexity of A.

This is surprising because Gaussian largeness is not the same thing as geometric largeness. In high dimensions, Gaussian measure concentrates near a thin annulus, and a set of Gaussian measure $3/4$ can be highly non-convex, full of holes, and geometrically very irregular. Talagrand's question asks whether repeated addition washes out this irregularity quickly enough to force the appearance of a genuinely convex body of Gaussian measure at least $1/2$.

Talagrand observed that the particular constants are not the essential issue. If the answer is positive with the assumption $\gamma_d(A) \geq 3/4$, then, after changing the universal number of summands, it remains positive under the assumption

$$\gamma_d(A) \geq \alpha$$

for any fixed $\alpha > 1/2$. Thus the problem is really about whether a set that is merely more likely than not, in Gaussian measure, must generate a large convex set after boundedly many additions. Talagrand also proved that $n = 2$ cannot work (Talagrand 1995); in fact, the paper recalls the stronger obstruction that two summands are insufficient even if one allows a universal rescaling of the convex body. The difficulty was to decide whether any bounded n works at all.

Hua, Song, and Tudose answer this question affirmatively. In the notation above, their result implies that there is a universal integer n such that, for every d and every balanced $A \subseteq \mathbb{R}^d$ with Gaussian measure bounded away from $1/2$, the set nA contains a convex body C with

$$\gamma_d(C) \geq \frac{1}{2}.$$

The paper formulates the geometric problem in an equivalent closed-set form, using a universal number of Minkowski sums and the threshold $2/3$, and notes that Talagrand's balanced formulation is equivalent to that version. One of the explicit geometric consequences proved in the paper is the following: there exists $\varepsilon > 0$ such that, whenever $A \subseteq \mathbb{R}^d$ is closed and

$$\gamma_d(A) \geq 1 - \varepsilon,$$

there is a convex body $K \subseteq \mathbb{R}^d$ such that

$$\gamma_d(K) \geq \frac{1}{2} \qquad \text{and} \qquad K \subseteq \varepsilon^{-1}(A + A + A).$$

For balanced sets, a dilation can be absorbed into a bounded number of Minkowski summands: if $\lambda \geq 1$ and $m \geq \lambda$ is an integer, then

$$\lambda A \subseteq mA.$$

Indeed, for $a \in A$, write λa as a sum of $\lfloor \lambda \rfloor$ copies of a plus one remaining multiple ta with $0 \leq t \leq 1$, and use balancedness. Thus a containment of the form

$$K \subseteq \varepsilon^{-1}(A + A + A)$$

is, in the balanced setting, a containment in nA for some universal n.

In short, the answer to Talagrand's question is yes. A large balanced set in Gaussian space need not look convex, and two additions are not enough to force large convex structure. Nevertheless, after a universal number of Minkowski additions, independent of the dimension, a convex body of Gaussian measure at least $1/2$ must appear. The key insight of Hua, Song, and Tudose is that this geometric phenomenon is governed by an apparently different but more tractable probabilistic fact: subgaussian random vectors are, up to universal constants, sums of a bounded number of coupled standard Gaussian vectors.

References

Hua, Dongming Merrick, Antoine Song, and Stefan Tudose. 2026. "On Talagrand's Convexity Conjecture." *arXiv Preprint arXiv:2605.10908*.

Talagrand, M. 1995. "Are All Sets of Positive Measure Essentially Convex?" In *Geometric Aspects of Functional Analysis (Israel, 1992–1994)*, vol. 77. Oper. Theory Adv. Appl. Birkhäuser, Basel.