Perfect powers in the Fibonacci sequence

Bogdan Grechuk • 29 May 2026

This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see this post.

My choice for 2003 in number theory is the theorem of Bugeaud, Mignotte, and Siksek determining all perfect powers in the Fibonacci sequence. Their result, proved in 2003 and published in 2006 (Bugeaud et al. 2006), resolved a long-standing open problem.

One of the most famous integer sequences in mathematics is the *Fibonacci sequence* $F_0, F_1, \dots, F_n, \dots$, defined by

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0. \tag{1}$$

The sequence has been studied so extensively that there is an entire journal, *The Fibonacci Quarterly*, devoted to its properties. Yet even for such a classical object, some very natural questions have either remained open for a long time or have been settled only recently. One such question asks: which Fibonacci numbers are perfect powers?

Some partial answers had been known for decades. A theorem of Ljunggren (Ljunggren 1951) implies that the only squares in the Fibonacci sequence are

$$0 = 0^2, \qquad 1 = 1^2, \qquad 144 = 12^2.$$

This result was rediscovered by Cohn (Cohn 1964) in 1964. In 1969, London and Finkelstein (London and Finkelstein 1969) proved that the only cubes in the sequence are

$$0, \qquad 1, \qquad 8 = 2^3.$$

The remaining problem was to decide whether any higher perfect powers occur. By the time of Bugeaud, Mignotte, and Siksek's work, it was already known that there were no further p-th powers for $p \leq 17$ or for $p \geq 5.1 \cdot 10^{17}$. Their theorem closed the enormous gap between these two ranges.

Theorem 1 *The only perfect powers in the Fibonacci sequence are*

$$F_0 = 0, \qquad F_1 = 1, \qquad F_2 = 1, \qquad F_6 = 8, \qquad F_{12} = 144.$$

The theorem is striking because the Fibonacci sequence is so elementary to define, while the proof draws on deep modern methods. It combines classical techniques from the theory of exponential Diophantine equations with the modular approach that emerged from the proof of Fermat's Last Theorem.

The same paper also treats the closely related Lucas sequence. This sequence $L_0, L_1, \ldots, L_n, \ldots$ is defined by

$$L_0 = 2, \qquad L_1 = 1, \qquad L_{n+2} = L_{n+1} + L_n \quad \text{for } n \geq 0.$$

Bugeaud, Mignotte, and Siksek prove that the only perfect powers in the Lucas sequence are $L_1 = 1$ and $L_3 = 4$. Thus the paper gives a complete answer not only for the Fibonacci sequence itself, but also for its most classical companion sequence.

References

Bugeaud, Yann, Maurice Mignotte, and Samir Siksek. 2006. "Classical and Modular Approaches to Exponential Diophantine Equations I. Fibonacci and Lucas Perfect Powers." *Ann. of Math.* 163 (3): 969–1018.

Cohn, J. H. E. 1964. "On Square Fibonacci Numbers." *J. Lond. Math. Soc.* 1 (1): 537–37.

Ljunggren, Wilhelm. 1951. "On the Diophantine Equation $x^2 + 4 = Ay^2$." *Norske Vid. Selsk. Forh., Trondheim* 24: 82–84.

London, Hymie, and Raphael Finkelstein. 1969. "On Fibonacci and Lucas Numbers Which Are Perfect Powers." *Fibonacci Quart.* 7 (5): 476–81.