Erdős’ unit distance conjecture disproved by AI

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In this previous blof post I described the resolution by AI of a problem about primitive sets of integers. Now AI made a substancial step further and disproved famous unit distance conjecture of Paul Erdős. This was one of the central open problem in discrete geometry, and its resolution deserves publication in a jounral of the highest level, such as Annals of Mathematics.

For a finite set P of points in the plane, let $\nu(P)$ denote the number of unordered pairs $\{A, B\} \subset P$ such that $d(A, B) = 1$. Let

$$U(n) = \max_{|P|=n} \nu(P).$$

Equivalently, since we may rescale the whole configuration, $U(n)$ is the maximum possible number of occurrences of the same distance among n points in the plane.

In 1946, Erdős (Erdős 1946) asked how large $U(n)$ can be. He observed that $U(n) = O(n^{3/2})$. This follows from the fact that the unit distance graph cannot contain a copy of $K_{2,3}$, since two unit circles meet in at most two points. In 1984, Spencer, Szemerédi and Trotter (Spencer et al. 1984) improved this to $U(n) = O(n^{4/3})$, and this remains the state of the art for the upper bound.

On the other hand, Erdős showed that the square grid construction gives

$$U(n) \geq n^{1+c/\log\log n},$$

for some $c > 0$ and infinitely many values of n. To achieve this, one takes a $k \times k$ grid with $k \approx \sqrt{n}$ and chooses an integer m with many representations as a sum of two squares. Many pairs of grid points then have distance $\sqrt{m}$, and after rescaling these become unit distances. Erdősa]$\mathrm{Erdos, P.@Erdős, P.}$ conjectured that the grid construction is close to the truth, namely that $U(n) \leq n^{1+o(1)}$.

In 2026, a counterexample to this conjecture was found by an internal model at OpenAI. A human-verified and simplified version of the proof was then presented in (Alon et al. 2026).

Theorem 1 *There exists a constant $\delta > 0$ such that*

$$U(n) \geq n^{1+\delta}$$

for infinitely many values of n.

The main idea of the construction is a striking generalization of Erdős's original grid example. The ordinary square grid may be viewed as a box in the Gaussian integers $\mathbb{Z}[i]$, where distances are controlled by representations of integers as sums of two squares. The new construction replaces the Gaussian integers by rings of integers $\mathcal{O}_K$ in suitable number fields K. The crucial additional idea is that the fields K are allowed to vary.

The proof in (Alon et al. 2026) made no attempts to optimize δ. Sawin (Sawin 2026) proved that one can take $\delta = 0.014$. This still leaves a large gap between the lower bound and the best known upper bound $U(n) = O(n^{4/3})$.

References

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