All triangulations have a common stellar subdivision

Bogdan Grechuk • 18 May 2026

In a recent paper (Adiprasito and Pak 2024), accepted by *Inventiones Mathematicae*, Adiprasito and Pak proved a striking compatibility theorem for triangulations: any two triangulations of the same geometric complex have a common stellar subdivision.

Suppose that the same geometric object is triangulated in two different ways. How are the two triangulations related?

Recall first that a d-*simplex* T in $\mathbb{R}^n$ is the convex hull of a set of $d+1$ affinely independent points. Thus T is a d-dimensional polytope with $d+1$ vertices. The convex hull of any non-empty subset of these vertices is called a *face* of T. A *simplicial complex* K is a collection of simplexes with two closure properties: every face of a simplex in K also belongs to K, and the non-empty intersection of any two simplexes in K is a face of both.

For a simplicial complex K in Euclidean space, let $|K|$ denote the union of all simplexes of K. More generally, a *geometric complex* Q in $\mathbb{R}^d$ is a finite collection of convex polytopes such that every face of a polytope in Q also belongs to Q, and the intersection of any two polytopes in Q is either empty or a common face of both. Its underlying set is

$$|Q| = \bigcup_{P \in Q} P.$$

A *triangulation* of Q is a simplicial complex A in $\mathbb{R}^d$ such that $|A| = |Q|$ and every simplex of A is contained in some polytope of Q. In other words, a triangulation cuts the polytopes of Q into simplexes without changing the underlying geometric set.

Now let A be a triangulation of Q, and let $z \in |A|$. There is a unique simplex σ of A whose relative interior contains z. The *stellar subdivision* of A at z is obtained by inserting z as a new vertex and replacing every simplex τ with $\sigma \subseteq \tau$ by the simplexes

$$\mathrm{conv}(\{z\} \cup \rho),$$

where conv denotes the convex hull and ρ ranges over the faces of τ that do not contain σ. All simplexes that do not contain σ are left unchanged. In dimension 2, this operation is easy to visualize: one may place a new vertex inside a triangle and divide the triangle into three smaller triangles, or place a new vertex on an edge and divide every triangle incident to that edge into two triangles.

A triangulation C of Q is called a *common stellar subdivision* of two triangulations A and B if C can be obtained from A by a finite sequence of stellar subdivisions and also from B by a finite sequence of stellar subdivisions.

For example, triangulate a square by one diagonal, and then triangulate the same square by the other diagonal. These two triangulations are different, but they have an obvious common stellar subdivision: insert a new vertex at the intersection of the two diagonals and join it to the four vertices of the square. The theorem of Adiprasito and Pak (Adiprasito and Pak 2024) says that this phenomenon is not special to polygons or to two-dimensional pictures.

Theorem 1 *Let Q be a geometric complex in $\mathbb{R}^d$, and let A and B be two triangulations of Q. Then A and B have a common stellar subdivision.*

At first sight, Theorem 1 may sound almost inevitable. If two triangulations cut up the same set, why not simply cut everything into smaller pieces until the two pictures agree? The subtle point is that the allowed cuts are severely restricted. We are not allowed to use arbitrary refinements. We must use only stellar subdivisions, one after another, starting separately from A and from B. In particular, we never delete a vertex, collapse an edge, or undo a previous subdivision. Both triangulations must move “forward” to the same final triangulation.

This is why Theorem 1 is a strong compatibility statement. It says that, for geometric complexes in Euclidean space, any two triangulations admit a common refinement reachable from each of them by the most elementary local refinement operation.

References

Adiprasito, Karim, and Igor Pak. 2024. “All Triangulations Have a Common Stellar Subdivision.” *arXiv Preprint arXiv:2404.05930*.