A lower bound for the non-diagonal Ramsey numbers

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In a recent paper (Ma et al. 2026), accepted by *Inventiones Mathematicae*, Ma, Shen and Xie proved the first exponential improvement over Erdős's classical lower bound for the off-diagonal Ramsey numbers $r(k, Ck)$, for every fixed constant $C > 1$.

Recall that, for positive integers k and m, the Ramsey number $r(k, m)$ is the smallest integer n such that every red–blue colouring of the edges of the complete graph K_n contains either a red copy of K_k or a blue copy of K_m. When $k = m$, the number $r(k, k)$ is called a diagonal Ramsey number; when $k \neq m$, it is called non-diagonal, or off-diagonal.

In a previous blog post, we discussed Ramsey numbers and some recent progress on upper bounds. In particular, the work (Campos et al. 2026) proves that there exists an absolute constant $\epsilon > 0$ such that

$$r(k, k) \leq (4 - \epsilon)^k$$

for all sufficiently large k. This gives an exponential improvement over the classical Erdős–Szekeres upper bound of order 4^k.

The same work also obtains a non-diagonal improvement: there exists a constant $\delta > 0$ such that

$$r(m, k) \leq \mathrm{e}^{-\delta m + o(k)} \binom{k+m}{m}$$

for all positive integers $m \leq k$. Later, Gupta (Gupta et al. 2024) showed that one may take $\delta = 0.05$. These results demonstrate that the classical upper bounds for Ramsey numbers are not merely improvable by polynomial factors, but by genuinely exponential factors in suitable ranges.

The situation for lower bounds has historically been more rigid. Erdős's classical probabilistic argument (Erdős 1947) shows that, for every fixed $C \geq 1$,

$$r(k, Ck) \geq ck \left(p_C^{-1/2}\right)^k, \tag{1}$$

where $c > 0$ is a constant and $p_C \in (0, 1/2]$ is the unique solution of

$$C = \frac{\log p_C}{\log(1 - p_C)}.$$

Here p_C is the edge-density parameter that optimizes the standard random-colouring construction. Roughly speaking, one colours each edge red with probability p_C and blue with probability $1 - p_C$, and chooses p_C so that the expected numbers of red K_k's and blue K_{Ck}'s are balanced. This produces the base $p_C^{-1/2}$ in (1), up to the polynomial factor ck.

For decades, no exponential improvement over this lower bound was known in the range $r(k, Ck)$ with fixed $C > 1$. Ma, Shen and Xie (Ma et al. 2026) have now broken this barrier.

Theorem 1 *For every constant $C > 1$, there exist constants $\varepsilon = \varepsilon(C) > 0$ and $k_0 = k_0(C) > 0$ such that, for every integer $k \geq k_0$,*

$$r(k, Ck) \geq \left(p_C^{-1/2} + \varepsilon\right)^k.$$

The point of the theorem is not merely that the lower bound is slightly larger than Erdős's bound. The improvement is exponential in k: the base of the exponential is increased from $p_C^{-1/2}$ to $p_C^{-1/2} + \varepsilon$. In Ramsey theory, where many classical bounds differ by exponential factors and where even small changes in the exponential base are highly significant, this is a substantial strengthening.

Conceptually, the result shows that the classical random-colouring construction is not optimal even at the exponential scale for any fixed non-diagonal ratio $C > 1$. This is especially striking because the Erdős lower bound had long served as the benchmark construction in this regime. The new theorem therefore changes the picture for off-diagonal Ramsey numbers: the natural probabilistic lower bound can be beaten uniformly throughout the entire range $r(k, Ck)$ with $C > 1$ fixed.

There remain many natural questions. The theorem guarantees the existence of an improvement $\varepsilon(C) > 0$, but determining the optimal exponential base for $r(k, Ck)$ remains far out of reach. Another open question is to extend Theorem 1 to the diagonal case $C = 1$.

References

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