The growth of diffusion in the 2D ASEP

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This post continues my series on favorite theorems of the 21st century. For an overview of the categories and earlier selections, see this post.

My choice for 2002 is Yau's theorem confirming the $(\log t)^{2/3}$ law for the two-dimensional asymmetric simple exclusion process (ASEP). The theorem was proved in 2002 and published in 2004 (Yau 2004).

The asymmetric simple exclusion process is a basic stochastic model of driven transport with interaction. Let us begin with the one-dimensional version. At time $t = 0$, each site $x \in \mathbb{Z}$ is occupied by a particle independently with probability σ. Each particle then waits for an exponential time of rate 1 and attempts to jump to a neighboring site: to the right with probability p and to the left with probability $q = 1 - p$. The jump is performed only if the target site is vacant; otherwise the particle remains where it is. The same rule is then repeated independently for all particles. This defines the one-dimensional asymmetric simple exclusion process, or ASEP.

For $x \in \mathbb{R}$, write $[x]$ for the integer part rounded toward zero: namely, the largest integer in $[0, x]$ if $x \geq 0$, and the smallest integer in $[x, 0]$ if $x \leq 0$. For a velocity $v \in \mathbb{R}$, define

$$J^{(v)}(t) = J_+^{(v)}(t) - J_-^{(v)}(t),$$

where $J_+^{(v)}(t)$ is the number of particles that started in $(-\infty, 0]$ at time 0 and are in $[[vt] + 1, \infty)$ at time t, while $J_-^{(v)}(t)$ is the number of particles that started in $[1, \infty)$ at time 0 and are in $(-\infty, [vt]]$ at time t. Thus $J^{(v)}(t)$ is the net particle current observed along the line moving with speed v during the time interval $[0, t]$.

The mean of $J^{(v)}(t)$ is relatively easy to estimate, but its variance is much subtler. In 1994, Ferrari and Fontes (Ferrari and Fontes 1994) proved that

$$\lim_{t\to\infty} \frac{\mathrm{Var}(J^{(v)}(t))}{t} = \sigma(1-\sigma)\left|(p-q)(1-2\sigma) - v\right|.$$

This shows that the variance grows linearly in time except at the velocity

$$v = (p - q)(1 - 2\sigma),$$

the characteristic speed. The corresponding random variable $J^{(v)}(t)$ is called the current across the characteristic, and for this special value of v the limiting formula above gives no information about the true order of growth. The problem remained open until 2010, when Balázs and Seppäläinen (Balázs and Seppäläinen 2010) proved that the variance grows like $t^{2/3}$. More precisely, if $v = (p-q)(1-2\sigma)$, then there are positive constants C and t_0 such that

$$C^{-1}t^{2/3} \leq \mathrm{Var}(J^{(v)}(t)) \leq Ct^{2/3}$$

for all $t \geq t_0$.

Now let us pass to the two-dimensional version. At time $t = 0$, each site $x \in \mathbb{Z}^2$ is occupied independently with probability σ. Each particle waits for an exponential time and then attempts to jump to one of its four neighboring sites, with prescribed probabilities p_1, p_2, p_3, p_4 for the four directions. As before, the jump is carried out only if the target site is unoccupied; otherwise the particle stays put. Repeating this rule independently for all particles gives the asymmetric simple exclusion process in the plane.

For concreteness, take

$$\sigma = \frac{1}{2}, \qquad p_1 = p_2 = \frac{1}{4}, \qquad p_3 = \frac{1}{2}, \qquad p_4 = 0.$$

For $x \in \mathbb{Z}^2$, let $\eta_x(t)$ be the occupation variable at site x and time t, so that $\eta_x(t)$ is either 0 or 1. Define

$$D(t) = \frac{4}{t} \sum_{x \in \mathbb{Z}^2} x_1^2 \operatorname{cov}\big(\eta_x(t), \eta_{(0,0)}(0)\big),$$

where x_1 denotes the first coordinate of x, and

$$\operatorname{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

is the covariance of two random variables. The function $D(t)$ is the diffusion coefficient. Its growth rate measures how quickly correlations spread in the driven system, and it is one of the central quantities in the study of superdiffusivity.

Yau proved that, in this two-dimensional asymmetric setting, $D(t)$ grows essentially like $(\log t)^{2/3}$ for large t. The precise statement is most naturally expressed through the Laplace transform of $tD(t)$.

Theorem 1 *There exists a constant $\gamma > 0$ such that, for all sufficiently small $\lambda > 0$,*

$$\lambda^{-2}|\log\lambda|^{2/3}\exp\big(-\gamma|\log\log\log\lambda|^2\big) \leq \int_0^\infty e^{-\lambda t} tD(t)\,dt \leq \lambda^{-2}|\log\lambda|^{2/3}\exp\big($$

The factor λ^{-2} is what one would expect from the Laplace transform of a function growing on the scale of $D(t)$, while the term $|\log\lambda|^{2/3}$ is the signature of the anomalous two-dimensional behavior. The additional factors

$$\exp\bigl(\pm\gamma|\log\log\log\lambda|^2\bigr)$$

are slowly varying corrections; they do not change the main exponent $2/3$.

The proof builds on earlier work of Landim, Quastel, Salmhofer, and Yau (Landim et al. 2004), which related the growth of $D(t)$ to estimates on the Green function of the dynamics. Their estimates reached degree three and implied, in particular, a lower bound of order $(\log t)^{1/2}$. Yau pushed the Green-function analysis much further and combined the resulting estimates with the framework of (Landim et al. 2004) to obtain Theorem 1. This confirmed the predicted $(\log t)^{2/3}$ law and made the theorem one of the landmark rigorous results on superdiffusivity in interacting particle systems.

References

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