The McKay conjecture on character degrees is true

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The May 2026 issue of the *Annals of Mathematics* contains a paper by Cabanes and Späth (Cabanes and Späth 2026) completing the proof of the celebrated McKay conjecture on character degrees. According to some specialists, this conjecture "lies at the heart of everything" in representation theory.

In a previous blog post, we discussed linear groups: groups whose elements can be represented by matrices in such a way that multiplication is preserved and distinct group elements correspond to distinct matrices. In this post, we relax the second requirement. We still represent group elements by matrices, but we no longer insist that different group elements give different matrices. For simplicity, all matrices below have complex entries.

Let G be a finite group. A *representation* of G is a map that assigns to each element $g \in G$ an invertible $k \times k$ complex matrix $\rho(g)$ such that

$$\rho(gh) = \rho(g)\rho(h), \qquad \text{for all } g, h \in G,$$

where the product on the left is the group operation in G, and the product on the right is matrix multiplication. A subspace $W \subset \mathbb{C}^k$ is called *invariant* under ρ if $\rho(g)v \in W$ for every $v \in W$ and every $g \in G$. The subspaces $\{0\}$ and $\mathbb{C}^k$ are always invariant; these are called the trivial invariant subspaces. A representation is called *irreducible* if it has no non-trivial invariant subspaces.

The map $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_\rho(g) = \mathrm{tr}(\rho(g)),$$

where the trace tr of a matrix is the sum of its diagonal entries, is called the *character* of the representation ρ. If ρ is irreducible, then χ_ρ is called an *irreducible character*. The size k of the matrices $\rho(g)$ is called the *degree* of the character. Equivalently, the degree of χ_ρ is $\chi_\rho(1)$, where 1 is the identity element of G.

Every finite group has only finitely many irreducible characters. It therefore makes sense to count irreducible characters with particular properties, such as those of odd or even degree. In 1972, McKay (McKay 1972) made a fundamental conjecture about the number of irreducible characters of odd degree in a finite group G. Denote this number by $m_2(G)$.

To state the conjecture, recall a few standard definitions. The *order* of a finite group G is the number of its elements. Let p be a prime, and let p^m be the largest power of p dividing $|G|$. A subgroup of G of order p^m is called a *Sylow p-subgroup*. If H is a subgroup of G, its *normalizer* in G is

$$N_G(H) = \{g \in G \mid g^{-1}Hg = H\}.$$

This is itself a subgroup of G, and it contains H. McKay conjectured that if H_2 is a Sylow 2-subgroup of G, then

$$m_2(G) = m_2(N_G(H_2)).$$

The conjecture arose from numerical observations about irreducible characters of small groups. It turned out, however, to be extraordinarily deep and became a major force in the development of finite group representation theory. Over time, researchers formulated stronger and more far-reaching versions of the conjecture, while even the original statement remained open for decades. Finally, after a long chain of partial results, Malle and Späth (Malle and Späth 2016) proved the original McKay conjecture in 2016.

Theorem 1 *Let G be a finite group, and let H_2 be a Sylow 2-subgroup of G. Then G and $N_G(H_2)$ have the same number of irreducible characters of odd degree.*

Although McKay (McKay 1972) originally formulated his conjecture for irreducible characters of odd degree, the conjecture was later generalized to arbitrary primes. For a finite group G and a prime p, one considers the irreducible characters of G whose degrees are not divisible by p. The general McKay conjecture predicts that the number of such characters of G is equal to the number of such characters of the normalizer of a Sylow p-subgroup of G.

In 2007, Isaacs, Malle, and Navarro (Isaacs et al. 2007) reduced this general conjecture to the verification of the so-called *inductive McKay condition* for a special class of finite groups called quasisimple groups. Malle and Späth (Malle and Späth 2016) proved Theorem 1 by verifying this condition for $p = 2$. In 2025, Späth (Späth 2025) verified it for $p = 3$, thereby proving the McKay conjecture in that case.

Theorem 2 *Let G be a finite group, and let H_3 be a Sylow 3-subgroup of G. Then G and $N_G(H_3)$ have the same number of irreducible characters of degree coprime to 3.*

For general primes p, verifying the inductive McKay condition proved even more difficult. A series of highly technical papers established it for many families of finite quasisimple groups. In theire recent Annals paper, Cabanes and Späth (Cabanes and Späth 2026) completed the remaining cases. Their work verifies the inductive McKay condition for the last outstanding families of finite quasisimple groups and, together with the reduction theorem of Isaacs, Malle, and Navarro, proves the McKay conjecture for all primes.

Theorem 3 *Let G be a finite group, let p be a prime, and let H_p be a Sylow p-subgroup of G. Then G and $N_G(H_p)$ have the same number of irreducible characters of degree coprime to p.*

Theorem 3 has many consequences, some of which are elementary to state but had resisted proof for decades. For example, it implies that if H_p is an abelian Sylow p-subgroup of G, then

$$k(G) \geq k(N_G(H_p)),$$

where $k(G)$ denotes the number of conjugacy classes of G.[1] This answers a question asked by Feit (Feit 1980) in 1980.

References

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1. Two elements a and b of a group G are called *conjugate* if $gag^{-1} = b$ for some $g \in G$. Conjugacy is an equivalence relation, and its equivalence classes are called *conjugacy classes*.↩