Rational approximations of linear subspaces

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The May 2026 issue of the *Annals of Mathematics* contains a paper by de Saxcé (Saxcé 2026) that determines Diophantine exponents and establishes Khintchine-type criteria for rational approximations of linear subspaces.

In two previous blog posts, here and here, we discussed Diophantine approximation and one of its foundational results: Khintchine's theorem. It states that if $\psi : \mathbb{N} \to \mathbb{R}^+$ is a function such that

$$\sum_{n=1}^{\infty} \psi(n) = \infty$$

and the sequence $n\psi(n)$ is non-increasing, then for almost every $\alpha \in \mathbb{R}$ there exist infinitely many pairs of integers m, n such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}. \tag{1}$$

Conversely, if

$$\sum_{n=1}^{\infty} \psi(n) < \infty,$$

then for almost every $\alpha \in \mathbb{R}$ there are at most finitely many pairs of integers m, n satisfying (1).

Approximation of an irrational number $\alpha \in \mathbb{R}$ by rational numbers m/n can be reformulated geometrically as approximation of the line $y = \alpha x$ through the origin by *rational lines*, that is, lines of the form

$$y = \frac{m}{n} x$$

for some rational number m/n. More formally, let L be the set of all lines in the plane passing through the origin. For two lines $l, l' \in L$, define their distance by

$$d(l, l') := \sin \angle(l, l'),$$

where $\angle(l, l')$ is the angle between them. Every rational line $l' \in L$ contains a unique pair of primitive integer points (n, m) and $(-n, -m)$, where m and n are coprime. Its height is then defined by

$$H(l') = \sqrt{n^2 + m^2}.$$

In this language, Hurwitz's theorem implies that, for every $\epsilon > 0$ and every non-rational line $l \in L$, there exist infinitely many rational lines $l' \in L$ such that

$$d(l, l') < \left(\frac{1}{\sqrt{5}} + \epsilon \right) H(l')^{-2}. \tag{2}$$

The exponent -2 and the constant $1/\sqrt{5}$ in this statement are best possible.

Similarly, simultaneous approximation of d irrational numbers $\alpha_1, \ldots, \alpha_d$ by rational numbers $r_1, \ldots, r_d$ can be reformulated as approximation of a line in $\mathbb{R}^{d+1}$. Namely, one considers the line l through the origin and the point

$$(1, \alpha_1, \ldots, \alpha_d),$$

and approximates it by rational lines in $\mathbb{R}^{d+1}$ through the origin, where a line is called rational if it contains at least two points with rational coordinates.

In 1967, Schmidt (Schmidt 1967) proposed a far-reaching generalization of this point of view. Instead of approximating lines by rational lines, he suggested studying the approximation of m-dimensional subspaces of $\mathbb{R}^d$ by rational k-dimensional subspaces of $\mathbb{R}^d$ of small height. To state this problem precisely, one has to define rational subspaces, their heights, and a distance between arbitrary subspaces.

A k-dimensional subspace $Y \subset \mathbb{R}^d$ is called rational if it has a basis $v_1, \ldots, v_k$ consisting of vectors with rational coordinates. Every rational subspace has a primitive basis: a basis $v_1, \ldots, v_k$ such that all v_i have integer coordinates and, if V is the $d \times k$ matrix with columns $v_1, \ldots, v_k$, then

$$\delta_k(V) = 1,$$

where

$$\delta_k(V) := \gcd \left\{ \det(M) \mid M \text{ is a } k \times k \text{ submatrix of } V \right\}.$$

Given a primitive basis $v_1, \ldots, v_k$ of Y, define

$$H(Y) := \sqrt{\det(G)},$$

where G is the $k \times k$ Gram matrix with entries

$$G_{ij} = \langle v_i, v_j \rangle.$$

Geometrically, $H(Y)$ is the k-dimensional volume of the parallelepiped spanned by $v_1, \ldots, v_k$. This quantity depends only on Y, not on the choice of primitive basis, and is called the height of Y.

For any subspace $X \subset \mathbb{R}^d$, let $L(X)$ denote the set of all lines through the origin contained in X. If X and Y are subspaces of $\mathbb{R}^d$ with $\dim X \leq \dim Y$, define

$$d(X,Y) := \max_{l_1 \in L(X)} \min_{l_2 \in L(Y)} \sin \angle(l_1, l_2). \tag{3}$$

If $\dim X > \dim Y$, define

$$d(X,Y) := d(Y,X),$$

where the right-hand side is given by (3).

With these definitions, Schmidt's theory of rational approximation to linear subspaces becomes a precise Diophantine problem. Let $\mathcal{X}_m(\mathbb{R}^d)$ be the set of all m-dimensional subspaces of $\mathbb{R}^d$, and let $\mathcal{X}_k(\mathbb{Q}^d)$ be the set of all rational k-dimensional subspaces of $\mathbb{R}^d$. For $X \in \mathcal{X}_m(\mathbb{R}^d)$ and $1 \le k \le d-1$, define the Diophantine exponent of X for approximation by rational k-dimensional subspaces by

$$\beta_k(X) := \inf\left\{\beta > 0 \;\middle|\; \exists c > 0 \text{ such that } \forall Y \in \mathcal{X}_k(\mathbb{Q}^d),\ d(X,Y) \ge cH(Y)^{-\beta}\right\}$$

The estimate (2) shows that $\beta_1(l) \ge 2$ for every $l \in \mathcal{X}_1(\mathbb{R}^2)$, and this lower bound is optimal.

More generally, Schmidt (Schmidt 1967) proved the optimal lower bound for $\beta_k(X)$, with $X \in \mathcal{X}_m(\mathbb{R}^d)$, in the case where the integers d, k, m satisfy

$$d \ge \min(k,m)\big(d - \max(k,m)\big).$$

For decades, however, no further cases were resolved. In 2026, de Saxcé (Saxcé 2026) proved the optimal lower bound for $\beta_k(X)$ in full generality, and in fact established a Khintchine-type theorem in this setting.

Theorem 1 *For all $X \in \mathcal{X}_m(\mathbb{R}^d)$ and all $1 \le k \le d-1$, one has*

$$\beta_k(X) \ge \frac{d}{g(k,m,d)},$$

where

$$g(k,m,d) = \min(k,m)\big(d - \max(k,m)\big).$$

Moreover, this lower bound is optimal.
More precisely, let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function, and define

$$I(\psi) = \int_1^{+\infty} \psi(u)^{g(k,m,d)} \frac{\mathrm{d}u}{u}.$$

Then the inequality

$$d(X,Y) \le H(Y)^{-d/g(k,m,d)} \psi(H(Y))$$

has only finitely many solutions $Y \in \mathcal{X}_k(\mathbb{Q}^d)$ for almost every $X \in \mathcal{X}_m(\mathbb{R}^d)$ if $I(\psi) < +\infty$, and has infinitely many such solutions for almost every $X \in \mathcal{X}_m(\mathbb{R}^d)$ if $I(\psi) = +\infty$.

In the same work, de Saxcé (Saxcé 2026) resolved several other fundamental problems in the theory of rational approximation to linear subspaces. In particular, he extended the approximation theory of submanifolds, discussed in a previous blog post, to the setting of linear subspaces. For these results, and for further developments, we refer the interested reader to the original paper (Saxcé 2026).

References

Saxcé, Nicolas de. 2026. “Rational Approximations to Linear Subspaces.” *Ann. Of Math. (2)* 203 (3): 737–80. https://doi.org/10.4007/annals.2026.203.3.2.

Schmidt, Wolfgang M. 1967. “On Heights of Algebraic Subspaces and Diophantine Approximations.” *Ann. Of Math. (2)* 85: 430–72. https://doi.org/10.2307/1970352.