A polygon whose billiard flow is weakly mixing

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The May 2026 issue of the *Annals of Mathematics* contains a paper by Chaika and Forni (Chaika and Forni 2026) proving the existence of a polygon whose billiard flow is weakly mixing.

A *billiard* in a polygon P is the motion of a point particle inside P with constant speed, subject to the usual reflection law: the angle of incidence equals the angle of reflection. A trajectory is called *equidistributed* if, for every measurable subset $M \subset P$ with area $S(M)$,

$$\lim_{T\to\infty} \frac{f_M(T)}{T} = \frac{S(M)}{S(P)},$$

where $S(P)$ is the area of P and $f_M(T)$ denotes the amount of time the trajectory spends in M during the interval $[0,T]$. A direction, parametrized by an angle $\alpha \in [0, 2\pi)$, is called *ergodic* if the corresponding billiard trajectory is equidistributed for almost every initial point; otherwise it is called *non-ergodic*.

A polygon Q is called *rational* if all of its angles, measured in radians, are rational multiples of π. A fundamental theorem of Kerckhoff, Masur, and Smillie (Kerckhoff et al. 1986a), proved in 1986, states that in every rational polygon almost every direction is ergodic.

Billiards provide one of the simplest and most vivid examples of a *continuous-time dynamical system*. Let $(X, \mathcal{B}, \mu)$ be a probability space. A *flow* is a one-parameter family $\{\varphi^t\}_{t\in\mathbb{R}}$ of measurable maps $\varphi^t : X \to X$ such that

$$\varphi^{t+s} = \varphi^t \circ \varphi^s \qquad \text{for all } t, s \in \mathbb{R}.$$

In particular, φ^0 is the identity map. The quadruple

$$\big(X, \mathcal{B}, \mu, \{\varphi^t\}_{t\in\mathbb{R}}\big)$$

is called a *continuous-time measure-preserving dynamical system* if

$$\mu\big((\varphi^t)^{-1}(A)\big) = \mu(A)$$

for every measurable set $A \in \mathcal{B}$ and every $t \in \mathbb{R}$. Such a system is called

1. *ergodic* if every invariant measurable set has measure 0 or 1, where $A \subset X$ is invariant if $(\varphi^t)^{-1}(A) = A$ for all $t \in \mathbb{R}$;

2. *weakly mixing* if, for every pair of measurable sets $A, B \subset X$,

3. $$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|\mu\big((\varphi^t)^{-1}(A) \cap B\big) - \mu(A)\mu(B)\right| \,\mathrm{d}t = 0;$$

   mixing if, for every pair of measurable sets $A, B \subset X$,

   $$\lim_{t\to\infty} \mu\big((\varphi^t)^{-1}(A) \cap B\big) = \mu(A)\mu(B).$$

These notions satisfy the implications

$$\text{mixing} \quad \Longrightarrow \quad \text{weakly mixing} \quad \Longrightarrow \quad \text{ergodic}.$$

If a direction θ is fixed, then billiards in a polygon P can be studied as a flow on P itself. Namely, one considers the family $\{\varphi_\theta^t\}_{t\in\mathbb{R}}$, where $\varphi_\theta^t(y)$ is the position at time t of a point whose initial position is $y \in P$ and whose initial direction is θ. When P is rational, the theorem of Kerckhoff, Masur, and Smillie says that this directional flow is ergodic for almost every θ.

A natural next question is when such directional flows are weakly mixing. In 2016, Avila and Delecroix (Avila and Delecroix 2016) answered this question for regular polygons: if $n \neq 3, 4, 6$, then the billiard flow in a regular n-gon is weakly mixing in almost every direction. More recently, Arana-Herrera, Chaika, and Forni (Arana-Herrera et al. 2024) gave a complete characterization of the rational polygons whose billiard flows are weakly mixing in almost every direction. It is known, however, that directional billiard flows in rational polygons cannot be mixing.

Now suppose that the direction is not fixed. The full billiard flow on a polygon P is naturally defined on the phase space X of pairs (y, u), where $y \in P$ is the position of the particle and $u \in \mathbb{R}^2$ is a unit vector giving its direction of motion. Boundary points are identified according to the reflection law: if y lies on a side s of P, then the incoming and outgoing unit vectors at y are identified whenever they are related by reflection across s. Thus the billiard flow is the family $\{\varphi^t\}_{t\in\mathbb{R}}$, where $\varphi^t(y, u)$ records both the position and the direction of the particle after time t. This flow preserves the Liouville measure, the natural uniform measure on phase space, normalized so that $\mu(X) = 1$.

Although almost every direction in a rational polygon is ergodic, the full billiard flow on a rational polygon cannot itself be ergodic, and hence cannot be weakly mixing or mixing. The reason is that the phase space decomposes into invariant pieces determined by direction: the flow moves within each such piece but does not mix different directions.

The situation changes dramatically for irrational polygons. In 1986, Kerckhoff, Masur, and Smillie (Kerckhoff et al. 1986b) proved that there exist irrational polygons whose full billiard flows are ergodic. This raised a stronger and long-standing question: can a polygonal billiard flow be weakly mixing? Chaika and Forni (Chaika and Forni 2026) have now answered this question affirmatively.

Theorem 1 *For every integer $n \geq 3$, there exists an n-vertex polygon P whose billiard flow is weakly mixing.*

In fact, Chaika and Forni prove more: for each fixed number of vertices, the set of polygons whose billiard flows are weakly mixing is dense in the appropriate space of polygons. Their result shows that weak mixing is not merely possible for polygonal billiards, but abundant in a topological sense.

Several fundamental questions remain open. It is not known whether there exists a polygon whose billiard flow is mixing. It is also unknown whether the billiard flow in every irrational polygon is ergodic, weakly mixing, or mixing.

References

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