The free-boundary Allen–Cahn problem

Bogdan Grechuk • 9 May 2026

In a paper (Chan et al. 2025) recently accepted for publication in the Journal of the American Mathematical Society (JAMC), Chan, Fernández-Real, Figalli and Serra solved the monotone free-boundary Allen–Cahn problem in $\mathbb{R}^4$.

This problem belongs to a broad family of questions in which one tries to understand whether simple qualitative assumptions force a solution to have a simple geometric shape. Such questions are central in elliptic partial differential equations, geometric analysis, and the calculus of variations. They ask, roughly speaking, whether a solution that is monotone in one direction must in fact depend on only one variable. In the classical Allen–Cahn equation, this is the content of De Giorgi's conjecture, which connects phase transition models with the geometry of minimal surfaces. The free-boundary version is especially interesting because the geometry is not prescribed in advance: the unknown function and the region in which it transitions between phases must be found simultaneously. Thus the problem combines analytic rigidity with the geometry of an unknown interface.

In the free-boundary Allen–Cahn problem, we consider a function $u : \mathbb{R}^n \to [-1, 1]$, which is allowed to be exactly equal to the two pure phases -1 and $+1$. The differential equation is imposed only in the region where the function is strictly between these two values. Let

$$T(u) = \{x \in \mathbb{R}^n \mid |u(x)| < 1\}.$$

We call $T(u)$ the *transition region*, and its boundary $\partial T(u)$ is called the *free boundary*. It is called "free" because it is not fixed in advance; it is part of what must be determined together with u.

The free-boundary Allen–Cahn problem is

$$\begin{cases} \Delta u = 0 & \text{in } T(u), \\ |\nabla u| = 1 & \text{on } \partial T(u), \end{cases} \tag{1}$$

where, as usual,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is the Laplacian, ∇ denotes the gradient, and $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^n$. A function satisfying $\Delta u = 0$ is called *harmonic*. Thus (1) says that u is harmonic inside the transition region, while on the free boundary its gradient has length 1.

We shall say that $u : \mathbb{R}^n \to [-1, 1]$ is a *global classical solution* of (1) if: (i) u is continuous on $\mathbb{R}^n$; (ii) u has continuous first and second partial derivatives inside $T(u)$; (iii) $T(u)$ has a smooth boundary, meaning that near every boundary point, after rotating the coordinate axes if necessary, the boundary is the graph of a smooth function of $n-1$ variables; (iv) the first partial derivatives of u have limiting values on $\partial T(u)$ when approached from inside $T(u)$; and (v) both equations in (1) hold pointwise.

The Hessian D^2u of a twice differentiable function $u : \mathbb{R}^n \to \mathbb{R}$ is the $n \times n$ matrix with entries

$$\frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad 1 \le i, j \le n.$$

Further, we say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is *one-dimensional* if there are a unit vector $e = (e_1, \ldots, e_n) \in \mathbb{R}^n$ and a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = \varphi(e_1 x_1 + \cdots + e_n x_n) \qquad \text{for every } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

In other words, u changes only in the single direction e and is constant along every hyperplane perpendicular to e.

In their recent JAMC paper, Chan, Fernández-Real, Figalli and Serra (Chan et al. 2025) proved the following result.

Theorem 1 *Let $u : \mathbb{R}^4 \to [-1, 1]$ be a global classical solution of the free-boundary Allen–Cahn problem (1). Suppose that*

$$\frac{\partial u}{\partial x_4}(x) > 0 \qquad \textit{for every } x \in T(u).$$

Then $D^2u \equiv 0$ in $T(u)$, and u is one-dimensional.

The assumption $\frac{\partial u}{\partial x_4} > 0$ says that, throughout the transition region, the solution is strictly increasing in one fixed direction. The theorem asserts that this apparently soft qualitative condition has a very strong consequence: the solution cannot have any genuinely four-dimensional structure.

The conclusion $D^2u \equiv 0$ in Theorem 1 means that every entry of the Hessian matrix is 0 at every point $x \in T(u)$. This is much stronger than $\Delta u = 0$: the Laplacian is only the sum of the diagonal entries of the Hessian. Thus harmonicity alone allows many complicated solutions, whereas the conclusion $D^2u \equiv 0$ says that u is affine inside each component of the transition region. In geometric terms, the free boundary is forced to be flat, and the solution is forced to transition only in one direction.

By adding dummy variables, we immediately conclude that Theorem 1 remains correct in all dimensions $n \leq 4$. On the other hand, its statement is known to fail in dimensions $n \geq 9$; see (Kamburov 2013). The problem remains open in dimensions $5 \leq n \leq 8$.

The result of Chan, Fernández-Real, Figalli and Serra is therefore a sharp new rigidity theorem in low dimension. It identifies dimension four as part of the range where monotonicity still enforces flatness for the free-boundary Allen–Cahn problem, while also fitting into the broader pattern suggested by De Giorgi-type problems: low-dimensional solutions are rigid, high-dimensional counterexamples exist, and the intermediate dimensions contain some of the most delicate open questions. In this sense, the theorem is not only a solution of a specific free-boundary problem in $\mathbb{R}^4$, but also an important step in understanding how phase-transition models, free boundaries, and geometric rigidity interact.

References

Chan, Hardy, Xavier Fernández-Real, Alessio Figalli, and Joaquim Serra. 2025. "Global Stable Solutions to the Free Boundary Allen–Cahn and Bernoulli Problems in 3D Are One-Dimensional." *arXiv Preprint arXiv:2503.21245*.

Kamburov, Nikola. 2013. "A Free Boundary Problem Inspired by a Conjecture of De Giorgi." *Comm. Partial Differential Equations* 38 (3): 477–528. https://doi.org/10.1080/03605302.2012.739672.