Efficient Steiner symmetrizations

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This post continues my series on favorite theorems of the 21st century. For an overview of the categories and earlier selections, see this post.

My choice for 2002 in Geometry and Topology is the theorem of Klartag and V. Milman saying that, after only $(2+\epsilon)n$ Steiner symmetrizations, any convex body in $\mathbb{R}^n$ can be made quantitatively close to a ball. The theorem has been proved in 2002 and published in 2003 (Klartag and Milman 2003).

One of the oldest classical problems in geometry asks for the maximal area A that can be enclosed by a closed curve of fixed length L. If the curve is a circle, then $A = L^2/4\pi$, and it was long conjectured that no other curve can enclose a larger area. Equivalently, among all plane curves enclosing a prescribed area, the circle has the least length. Steiner (Steiner 1838) proved this conjecture in 1838, under the additional assumption that a solution exists. Since then, many rigorous unconditional proofs have been found.

The same principle holds in higher dimensions: among all bodies of a given volume, the Euclidean ball has the least perimeter. This result is usually stated as the isoperimetric inequality

$$P(S) \geq nV(S)^{\frac{n-1}{n}} V(B_1)^{\frac{1}{n}},$$

where $S \subset \mathbb{R}^n$ is a measurable set, $P(S)$ is its perimeter, that is, the $(n-1)$-dimensional measure of its boundary in the appropriate sense, $V(\cdot)$ denotes n-dimensional volume, and B_1 is the unit ball in $\mathbb{R}^n$. In this generality, without unnecessary regularity assumptions on S, the inequality was proved by De Giorgi (De Giorgi 1958) in 1958.

In his proof of the planar isoperimetric inequality, Steiner introduced the powerful method of *symmetrization*. Since then, symmetrization has become one of the central tools for proving geometric inequalities. Let $0 \neq h \in \mathbb{R}^n$, and let

$$H = \{x \in \mathbb{R}^n : \langle x, h \rangle = 0\}$$

be the hyperplane through the origin perpendicular to h. Every point $x \in \mathbb{R}^n$ can be written uniquely as $x = y + th$, with $y \in H$ and $t \in \mathbb{R}$; we shall regard (y, t) as coordinates adapted to the direction h. For fixed y, the set $y + \mathbb{R}h$ is the line through y in the direction h.

The *Steiner symmetrization* of a convex body K with respect to H is obtained by replacing every chord of K parallel to h by the centered interval of the same length. In formulas,

$$S_H(K) = \left\{ (y,t) : K \cap (y + \mathbb{R}h) \neq \emptyset,\ |t| \leq \frac{1}{2} |K \cap (y + \mathbb{R}h)| \right\},$$

where $|K \cap (y + \mathbb{R}h)|$ denotes the length of the interval $K \cap (y + \mathbb{R}h)$.

In 1909, Carathéodory and Study (Carathéodory and Study 1909) proved that, for every convex body K, there exists a sequence of Steiner symmetrizations that converges to a ball. This gives a remarkably flexible strategy for proving statements about convex bodies. Suppose a property P has the following two features: first, whenever P holds for $S_H(K)$, it also holds for K; and second, P holds for balls and for bodies sufficiently close to balls. Then P holds for all convex bodies. Indeed, by the Carathéodory–Study theorem, after finitely many symmetrizations one obtains a body

$$K' = S_m(\cdots S_2(S_1(K)) \cdots)$$

which is close enough to a ball. The property P holds for K' by the second assumption, and then, by applying the first assumption m times, it follows that P holds for the original body K.

This is a very general and powerful method. Its quantitative version, however, is more delicate. If P is a quantitative property involving parameters, then the implication from $S_H(K)$ back to K may come with a loss in those parameters. Each symmetrization may therefore cost something. In such arguments, it becomes essential to know how many symmetrizations are actually needed in order to bring an arbitrary convex body close to a ball.

The first general bounds were very large. In 1951, Hadwiger (Hadwiger 1951) proved an upper bound of order $n^{n/2}$, where n is the dimension. In 1989, Bourgain, Lindenstrauss, and V. Milman (Bourgain et al. 1989) made a dramatic improvement, proving that one can transform any convex body into a body moderately close to a ball using only $cn \log n$ Steiner symmetrizations. Klartag and V. Milman (Klartag and Milman 2003) then reduced this to a linear number of steps.

Theorem 1 *For every $\epsilon > 0$, there exists a constant $c > 0$ such that, for every $n \geq 2$ and every convex body $K \subset \mathbb{R}^n$, there are $m \leq (2+\epsilon)n$ Steiner symmetrizations $S_1, S_2, \ldots, S_m$ for which the body*

$$K' = S_m(S_{m-1}(\cdots S_2(S_1(K))\cdots))$$

contains a Euclidean ball of radius r_1 and is contained in a Euclidean ball of radius r_2, with

$$\frac{r_1}{r_2} = c.$$

Thus, after only slightly more than $2n$ symmetrizations, an arbitrary convex body can be made *isomorphic* to a ball: not necessarily close in the fine metric sense, but comparable to a ball up to a universal constant factor.

One striking feature of the proof is that the authors first reduce the general problem to two extremal-looking special cases: the cube and the cross-polytope, the convex hull of the points obtained by permuting the coordinates of $(\pm 1, 0, \ldots, 0)$. The cube case is comparatively easy. The cross-polytope case is subtler, but Klartag and Milman handle it by an elementary argument. Proving merely that $m \leq Cn$ symmetrizations suffice already follows from this reduction. To obtain the sharp constant $2+\epsilon$, however, they use deeper tools, including Milman's quotient-of-subspace theorem.

The result is a beautiful example of the modern asymptotic theory of convex bodies: a classical geometric operation, introduced by Steiner in the nineteenth century for the isoperimetric problem, is combined with high-dimensional convex geometry to produce an almost optimal linear bound.

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