On the symplectic nonsqueezing

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In a recent paper accepted by *Acta Mathematica*, McDuff and Siegel (McDuff and Siegel 2024) computed, for every stabilized four-dimensional ellipsoid, the smallest stabilized four-ball into which it can be symplectically embedded. Their result settles a major open problem in symplectic geometry.

Symplectic geometry is an even-dimensional geometry that measures the signed areas of two-dimensional objects. To begin with a simple case, let $S \subset \mathbb{R}^2$ be an open region in the plane, together with a choice of orientation, that is, a direction in which to traverse its boundary ∂S. The *symplectic area* $\omega(S)$ is the real number whose absolute value $|\omega(S)|$ is the usual Euclidean area of S, and whose sign is positive if the orientation is anticlockwise and negative if it is clockwise.

More generally, if S is an oriented two-dimensional surface in $\mathbb{R}^{2n}$, we define its symplectic area by projecting it to each coordinate plane (x_{2i-1}, x_{2i}) and summing the signed areas:

$$\omega(S) = \sum_{i=1}^{n} \omega(S_i),$$

where S_i denotes the projection of S to the (x_{2i-1}, x_{2i})-plane. If $U, V \subset \mathbb{R}^{2n}$, we say that U *embeds symplectically* into V, and write

$$U \overset{s}{\hookrightarrow} V,$$

if there is a smooth embedding $\phi : U \to \mathbb{R}^{2n}$ such that (i) $\phi(U) \subset V$, (ii) ϕ is a diffeomorphism from U onto its image, and (iii) ϕ preserves symplectic area:

$$\omega(S) = \omega(\phi(S))$$

for every oriented surface $S \subset U$.

Questions about symplectic embeddings lie at the heart of symplectic geometry. For instance, Gromov's celebrated non-squeezing theorem says that the Euclidean ball

$$B^{2n}(r) := \left\{ x \in \mathbb{R}^{2n} : \sum_{i=1}^{2n} x_i^2 \le r^2 \right\}$$

of radius $r > 0$ embeds symplectically into the cylinder

$$Z^{2n}(R) := B^2(R) \times \mathbb{R}^{2(n-1)} = \left\{x \in \mathbb{R}^{2n} : x_1^2 + x_2^2 \leq R^2\right\} \tag{1}$$

if and only if $r \leq R$. Thus, although the cylinder is unbounded in all but two directions, the two-dimensional radius of its base still imposes a rigid obstruction to symplectic embeddings.

One consequence is that if the symplectic *polydisk*

$$P(R_1, \ldots, R_n) := B^2(R_1) \times \cdots \times B^2(R_n),$$

with positive radii $R_1, \ldots, R_n$, embeds symplectically into another polydisk

$$P(R'_1, \ldots, R'_n),$$

then

$$\min_i R_i \leq \min_i R'_i.$$

Conservation of volume gives another necessary condition:

$$\prod_{i=1}^{n} R_i \leq \prod_{i=1}^{n} R'_i.$$

In 2008, Guth (Guth 2008) proved that, up to a constant factor depending only on the dimension, these obvious necessary conditions are also sufficient.

Theorem 1 *For every $n \geq 1$, there is a constant $C_n > 0$ such that if*

$$C_n \min_i R_i \leq \min_i R'_i \qquad \text{and} \qquad C_n \prod_{i=1}^{n} R_i \leq \prod_{i=1}^{n} R'_i,$$

then

$$P(R_1, \ldots, R_n) \overset{s}{\hookrightarrow} P(R'_1, \ldots, R'_n).$$

Theorem 1 has many striking consequences. For example, it implies that there is a constant R such that

$$B^2(1) \times B^{2(n-1)}(S) \overset{s}{\hookrightarrow} B^4(R) \times \mathbb{R}^{2(n-2)}$$

for every $S > 0$. In 2014, Hind and Kerman (Hind and Kerman 2014) proved that the infimum of all such constants R is exactly $\sqrt{3}$.

Theorem 2 *Fix $n \geq 3$. For every $R > \sqrt{3}$,*

$$B^2(1) \times B^{2(n-1)}(S) \overset{s}{\hookrightarrow} B^4(R) \times \mathbb{R}^{2(n-2)}$$

for all $S > 0$. Conversely, for every $0 < R < \sqrt{3}$, there is a constant $S_0 = S_0(R, n)$ such that, for all $S > S_0$, there is no symplectic embedding

$$B^2(1) \times B^{2(n-1)}(S) \overset{s}{\hookrightarrow} B^4(R) \times \mathbb{R}^{2(n-2)}.$$

Theorems 1 and 2 concern balls and products of balls, which are among the most tractable examples in symplectic geometry. Another natural class of domains is given by ellipsoids. For a real number $a \geq 1$, define the four-dimensional symplectic ellipsoid

$$E_a := \left\{ x \in \mathbb{R}^4 : x_1^2 + x_2^2 + \frac{x_3^2 + x_4^2}{a} \leq 1 \right\}.$$

Let $c_{\text{ell}}(a)$ denote the infimum of all $\mu > 0$ such that

$$E_a \overset{s}{\hookrightarrow} B^4(\sqrt{\mu}).$$

Equivalently, $c_{\text{ell}}(a)$ measures how much the radius of the target ball must be enlarged, in squared-radius units, in order to contain a symplectic image of E_a. Conservation of volume implies

$$c_{\text{ell}}(a) \geq \sqrt{a}$$

for all $a \geq 1$. It is also not hard to see that $c_{\text{ell}}(a)$ is nondecreasing and continuous. The exact computation of this function was a long-standing problem, solved by McDuff and Schlenk (McDuff and Schlenk 2012) in 2012.

Let F_n, $n \geq 0$, be the Fibonacci numbers:

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n.$$

Set

$$\alpha_0 := 1, \qquad \beta_0 := 2,$$

and, for $k \geq 1$, define

$$\alpha_k := \frac{F_{2k+1}^2}{F_{2k-1}^2}, \qquad \beta_k := \frac{F_{2k+3}}{F_{2k-1}}.$$

Then

$$1 = \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \cdots ,$$

and these numbers converge to

$$\tau^4, \qquad \tau = \frac{1 + \sqrt{5}}{2}.$$

The graph of $c_{\text{ell}}(a)$ contains an infinite staircase on the interval $[1, \tau^4]$, now known as the Fibonacci staircase.

Theorem 3 *For every real number $a \geq 1$, the function $c_{\text{ell}}(a)$ has the following form.*

(i) For every $k \geq 0$,

$$c_{\text{ell}}(a) = \frac{a}{\sqrt{\alpha_k}} \quad \text{for } a \in [\alpha_k, \beta_k],$$

and

$$c_{\text{ell}}(a) = \sqrt{\alpha_{k+1}} \quad \text{for } a \in [\beta_k, \alpha_{k+1}].$$

(ii) For $\tau^4 \leq a \leq 7$,

$$c_{\text{ell}}(a) = \frac{a+1}{3}.$$

(iii) There are finitely many pairwise disjoint closed intervals

$$I_j \subset \left[7, 8\frac{1}{36}\right]$$

such that

$$c_{\text{ell}}(a) = \sqrt{a}$$

for all $a > 7$ with $a \notin I_j$ for every j. Moreover, on each interval I_j, the function $c_{\text{ell}}(a)$ is piecewise linear and has exactly one nonsmooth point in the interior of I_j.

(iv) For $a \geq 8\frac{1}{36}$,

$$c_{\text{ell}}(a) = \sqrt{a}.$$

For a positive integer n, the product

$$U \times \mathbb{R}^{2n}$$

is obtained from a four-dimensional region $U \subset \mathbb{R}^4$ by adjoining $2n$ extra unbounded real coordinates. This operation is called *stabilization*. The stabilized analogue of $c_{\text{ell}}(a)$ is

$$c_{B^4 \times \mathbb{R}^{2n}}(a) := \inf \left\{ \mu > 0 : E_a \times \mathbb{R}^{2n} \overset{s}{\hookrightarrow} B^4(\sqrt{\mu}) \times \mathbb{R}^{2n} \right\}.$$

Thus $c_{B^4 \times \mathbb{R}^{2n}}(a)$ measures how much the round four-ball factor must be enlarged before the stabilized ellipsoid embeds symplectically into it. Upper and lower bounds for this quantity, for various values of a, have been studied by many authors.

In their recent paper, McDuff and Siegel (McDuff and Siegel 2024) computed

$$c_{B^4 \times \mathbb{R}^{2n}}(a)$$

exactly for every $a \geq 1$. Their result gives a stabilized counterpart of Theorem 3.

Theorem 4 *For every positive integer n and every real number $a \geq 1$,*

$$c_{B^4\times\mathbb{R}^{2n}}(a) = \begin{cases} \dfrac{a}{\sqrt{\alpha_k}}, & \text{if } a \in [\alpha_k, \beta_k] \text{ for some } k \geq 0, \\ \sqrt{\alpha_{k+1}}, & \text{if } a \in [\beta_k, \alpha_{k+1}] \text{ for some } k \geq 0, \\ \dfrac{3a}{a+1}, & \text{if } a \in [\tau^4, \infty). \end{cases}$$

In particular,

$$c_{B^4\times\mathbb{R}^{2n}}(a) = c_{\text{ell}}(a)$$

for

$$1 \leq a \leq \tau^4.$$

That is, stabilization does not change the embedding function along the Fibonacci staircase. Beyond the staircase, however, the stabilized problem behaves very differently from the four-dimensional one. Instead of eventually becoming governed by the volume constraint

$$c_{\text{ell}}(a) = \sqrt{a},$$

the stabilized function is

$$c_{B^4\times\mathbb{R}^{2n}}(a) = \frac{3a}{a+1}$$

for all

$$a \geq \tau^4.$$

The previously known upper bound is therefore sharp. The new contribution of McDuff and Siegel (McDuff and Siegel 2024) is the matching lower bound for every $a \geq \tau^4$.

References

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