Grigorchuk's gap conjecture and simple groups

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In a recent preprint (Eberhard et al. 2026), Eberhard, Maini, Sabatini and Tracey reduce the celebrated Grigorchuk gap conjecture to two questions about simple groups.

Recall from yesterday's blog post that, if G is a finitely generated group and S is a finite generating set, the growth function $\gamma_S(n)$ counts the number of elements of G that can be written as a product of at most n elements of $S \cup S^{-1}$. We say that G has *polynomial growth* if there are constants $C, k < \infty$ such that

$$\gamma_S(n) \leq C(n^k + 1)$$

for every n. This property is independent of the choice of finite generating set S: changing the generating set can distort distances only by a bounded multiplicative factor, so the qualitative distinction between polynomial and superpolynomial growth is intrinsic to the group.

Grigorchuk's gap conjecture predicts that polynomial growth is separated from genuinely faster growth by a uniform gap. More precisely, it asserts that there exists a constant $\beta > 0$ such that no finitely generated group has growth which is superpolynomial but still bounded above by e^{n^β}. In other words, according to the conjecture, once the growth of a finitely generated group escapes every polynomial bound, it must grow at least as fast as a fixed stretched exponential function. This conjecture is one of the central open problems in the theory of groups of intermediate growth. It is especially striking because Grigorchuk's own examples show that there are finitely generated groups whose growth is faster than every polynomial but slower than every exponential function; the conjecture says that even within this intermediate regime, there should be a universal lower threshold.

The result of Eberhard, Maini, Sabatini and Tracey (Eberhard et al. 2026) shows that Grigorchuk's gap conjecture follows from two conjectural statements about simple groups. The first is the gap conjecture restricted to simple groups. The second is Babai's diameter conjecture, discussed in a previous blog post. Recall that Babai's conjecture predicts that the diameter of every non-abelian finite simple group is bounded above by a fixed polynomial in $\log |G|$. Here

$$\mathrm{diam}(G) = \max_S d(G, S),$$

where the maximum is taken over all generating sets S of G, and $d(G, S)$ is the smallest integer m such that every element $g \in G$ can be expressed as a product of at most m elements of $S \cup S^{-1}$.

This is a remarkable reduction. Grigorchuk's conjecture is a statement about all finitely generated groups, a class far too large to admit any classification. By contrast, the finite simple groups are classified, and infinite simple groups form a much more rigid and structured class than arbitrary finitely generated groups. Babai's conjecture is still very difficult, but it is a finite-group-theoretic statement about non-abelian finite simple groups. In principle, the classification theorem allows one to approach it family by family: alternating groups, groups of Lie type of bounded rank, groups of Lie type of unbounded rank, and the sporadic groups.

The philosophical point is that possible counterexamples to the gap conjecture cannot be completely arbitrary. The theorem of Eberhard, Maini, Sabatini and Tracey suggests that, assuming good diameter bounds for finite simple groups, any obstruction to a universal growth gap must already be visible in the world of simple groups. Thus a broad and notoriously difficult problem about finitely generated groups is transformed into a pair of more focused problems, both centered on simple groups. This does not solve Grigorchuk's conjecture, but it significantly clarifies where the real difficulty may lie.

References

Eberhard, Sean, Elena Maini, Luca Sabatini, and Gareth Tracey. 2026. "Diameter Bounds for Arbitrary Finite Groups and Applications." *arXiv Preprint arXiv:2604.15303*.