Groups of non-uniform exponential growth

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see this post.

My choice for 2002 in Algebra is John S. Wilson's construction of finitely generated groups which have exponential growth but fail to have uniform exponential growth. The result gives a negative answer to one of the central open questions in the theory of growth of groups, a question posed by Gromov in 1981 (Gromov 1981). Wilson's theorem was proved in 2002 and published in 2004 (Wilson 2004).

The point of the theorem is that there are two rather different ways in which a finitely generated group can grow exponentially. The first is qualitative: balls in the Cayley graph grow exponentially fast for one, and hence for every, finite generating set. The second is quantitative and uniform: there is a single exponential lower bound which works independently of the choice of finite generating set. Gromov's question asked whether the first phenomenon always forces the second. Wilson showed that it does not.

Recall that a *generating set* of a group G is a subset $S \subset G$ such that every element $g \in G$ can be written as a finite product of elements of S and their inverses. The group G is *finitely generated* if it admits a finite generating set. Given such a finite generating set S, let

$$\gamma_S(n) = \#\{g \in G : g \text{ can be written as a word of length at most } n \text{ in } S \cup S^{-}$$

Equivalently, $\gamma_S(n)$ is the size of the ball of radius n around the identity in the Cayley graph $\mathrm{Cay}(G, S)$.

The exponential growth rate of G with respect to S is

$$e_S(G) = \lim_{n \to \infty} \sqrt[n]{\gamma_S(n)}.$$

The existence of this limit follows from submultiplicativity: the inequality

$$\gamma_S(m+n) \leq \gamma_S(m)\gamma_S(n)$$

implies that the limiting exponential rate is well-defined.

A finitely generated group G is said to have *exponential growth* if $e_S(G) > 1$ for some finite generating set S. This condition is independent of the choice of finite generating set: if S and T are two finite generating sets, then each element of S has bounded word length with respect to T, and conversely. Thus the two corresponding word metrics are quasi-isometric, and the property of having exponential growth does not depend on the chosen generators. In particular, if $e_S(G) > 1$ for one finite generating set S, then $e_T(G) > 1$ for every finite generating set T.

Uniform exponential growth is the stronger requirement that the exponential growth rate be bounded away from 1 uniformly over all finite generating sets. That is, G has *uniform exponential growth* if

$$\inf_S e_S(G) > 1, \tag{1}$$

where the infimum is taken over all finite generating sets S of G. Thus a group has exponential growth but not uniform exponential growth precisely when every finite generating set gives exponential growth, but there are finite generating sets for which the growth rate is arbitrarily close to 1.

Gromov asked whether such behavior could occur. His question was natural partly because the known mechanisms producing exponential growth tended to produce it uniformly. For example, if a group contains a free semigroup on two generators with words of uniformly bounded length, then one obtains a uniform exponential lower bound. More generally, many classes of groups for which exponential growth was understood were eventually shown to have uniform exponential growth. A particularly important affirmative result is the case of finitely generated linear groups: if a finitely generated linear group has exponential growth, then it has uniform exponential growth. Results of this kind made Gromov's question seem plausible for a long time.

Wilson's theorem showed that the general picture is subtler.

Theorem 1 (Wilson) *There exists a finitely generated group G which has exponential growth but does not have uniform exponential growth.*

In other words, Wilson constructed a finitely generated group G such that

$$e_S(G) > 1$$

for every finite generating set S, but

$$\inf_S e_S(G) = 1.$$

Thus G grows exponentially no matter how one chooses generators, yet there are generating sets with respect to which the exponential growth is as slow as desired.

References

Gromov, Mikhael. 1981. *Structures métriques pour les variétés riemanniennes*. Vol. 1. Textes Mathématiques [Mathematical Texts]. Cedic.

Wilson, John S. 2004. "On Exponential Growth and Uniformly Exponential Growth for Groups." *Invent. Math.* 155 (2): 287–303.