Splitting spheres for links of 2-spheres

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In a recent paper (Tatsuoka 2026), accepted by *Inventiones Mathematicae*, Tatsuoka proved that the unlink of two unknotted 2-spheres in S^4 admits infinitely many non-isotopic splitting 3-spheres.

We have discussed knots and links in several previous blog posts; see, for example, this post. The same ideas can be studied in higher dimensions by changing the dimension of the objects being knotted. For $n \geq 0$, the n-sphere is

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\},$$

where $|x|$ denotes the Euclidean norm. A 2-*knot* is a smoothly embedded copy of S^2 in S^4. More generally, a 2-*link* is a finite disjoint union of smoothly embedded copies of S^2 in S^4. Thus a two-component 2-link has the form

$$K = K_1 \sqcup K_2,$$

where K_1 and K_2 are disjoint embedded 2-spheres in S^4.

A 2-knot $K \subset S^4$ is called *unknotted* if it bounds a smoothly embedded 3-ball in S^4. A two-component 2-link $K = K_1 \sqcup K_2$ is called the *unlink of two unknotted 2-spheres* if K_1 and K_2 bound disjoint smoothly embedded 3-balls in S^4.

For a 2-link $K \subset S^4$, let $\nu(K)$ denote a small open tubular neighbourhood of K. For a single embedded 2-sphere, this is a small open normal thickening, homeomorphic to $S^2 \times D^2_\varepsilon$, where

$$D^2_\varepsilon = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \varepsilon^2\}.$$

For a two-component link $K = K_1 \sqcup K_2$, the neighbourhood $\nu(K)$ is the union of two such disjoint neighbourhoods. The complement

$$S^4 \setminus \nu(K)$$

is called the *exterior* of the link.

A *splitting* 3-*sphere* for a two-component 2-link $K = K_1 \sqcup K_2$ is a smoothly embedded copy $\Sigma \cong S^3$ in the exterior $S^4 \setminus \nu(K)$ such that K_1 and K_2 lie in different connected components of $S^4 \setminus \Sigma$. Equivalently, every continuous path in S^4 from K_1 to K_2 must meet Σ. Thus Σ separates the two components of the link.

Two splitting 3-spheres Σ_0 and Σ_1 for the same link K are called *isotopic* if they are isotopic through splitting 3-spheres in the link exterior. More explicitly, this means that there is a continuous family of smooth embeddings

$$F_t : S^3 \to S^4 \setminus \nu(K), \qquad 0 \leq t \leq 1,$$

such that

$$F_0(S^3) = \Sigma_0, \qquad F_1(S^3) = \Sigma_1,$$

and each $F_t(S^3)$ is a splitting 3-sphere for K. A collection of splitting 3-spheres is called *pairwise non-isotopic* if no two distinct members of the collection are isotopic in this sense.

In classical knot theory, splitting spheres are rigid, in sense that for a two-component link in S^3, any two splitting 2-spheres are isotopic in the link exterior. It is therefore natural to ask whether an analogous uniqueness statement holds one dimension higher. Dimension 4, however, often supports phenomena with no direct analogue in dimension 3. Hughes, Kim and Miller (Hughes et al. 2025) showed that splitting 3-spheres need not be unique for certain links of surfaces in S^4, and they asked whether such non-uniqueness already appears for the simplest possible example: the unlink of two unknotted 2-spheres.

Tatsuoka answered this question affirmatively.

Theorem 1 *Let K be the unlink of two unknotted 2-spheres in S^4. Then K admits infinitely many pairwise non-isotopic splitting 3-spheres.*

Theorem 1 shows that even the most elementary two-component 2-link in S^4 has a surprisingly rich collection of separating 3-spheres in its exterior. The failure of uniqueness is not caused by knotting of the components: each component is an unknotted 2-sphere, and the two components are unlinked. Rather, the complexity comes from the many different ways in which 3-spheres can sit inside the complement $S^4 \setminus \nu(K)$ while the link itself is kept fixed.

References

Hughes, Mark, Seungwon Kim, and Maggie Miller. 2025. “Non-Isotopic Splitting Spheres for a Split Link in S4.” *Proceedings of the London Mathematical Society* 130 (4): e70038.

Tatsuoka, Alison. 2026. “Splitting 3-Spheres for Unknotted 2-Spheres in S4.” *Inventiones Mathematicae*, 1–8.