The existence of exotic aspherical 4-manifolds

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In a recent paper accepted for publication in the *Annals of Mathematics*, Davis, Hayden, Huang, Ruberman, and Sunukjian proved the existence of exotic pairs of smooth, closed, aspherical 4-manifolds (Davis et al. 2026).

Let X and Y be topological spaces. Two continuous maps $F, G : X \to Y$ are said to be *homotopic* if there exists a continuous map

$$H : X \times [0,1] \to Y$$

such that $H(x,0) = F(x)$ and $H(x,1) = G(x)$ for every $x \in X$. A continuous map $F : X \to Y$ is a *homotopy equivalence* if there is a continuous map $G : Y \to X$ such that $G \circ F$ is homotopic to the identity map on X, and $F \circ G$ is homotopic to the identity map on Y. In this case, G is called a homotopy inverse of F.

A path-connected topological k-manifold M is called *aspherical* if all of its higher homotopy groups vanish. This means that for every integer $i \geq 2$, every continuous map

$$\mathbb{S}^i \to M$$

from the i-sphere to M is homotopic to a constant map.

A topological manifold can sometimes support more than one smooth structure. We shall call two smooth manifolds M and N an *exotic pair* if they are homeomorphic but not diffeomorphic. Equivalently, after identifying M and N by a homeomorphism, one obtains two distinct smooth structures on the same underlying topological manifold, distinct up to diffeomorphism.

The *Borel conjecture* predicts that closed aspherical topological manifolds are topologically rigid: if M and N are closed aspherical topological manifolds and $f : M \to N$ is a homotopy equivalence, then f

should be homotopic to a homeomorphism. A natural smooth analogue asks whether two closed smooth aspherical n-manifolds that are homotopy equivalent must be diffeomorphic.

This smooth analogue is classical in dimensions $n \leq 2$. In dimension 3, it is known for orientable manifolds as a consequence of Perelman's proof of geometrization. In contrast, counterexamples had long been known in every dimension $n \geq 5$. Thus dimension 4 was the remaining case.

Davis, Hayden, Huang, Ruberman, and Sunukjian resolved this case by proving the following theorem.

Theorem 1 *There exist smooth, closed, aspherical 4-manifolds that are homeomorphic but not diffeomorphic.*

Theorem 1 shows that the topological rigidity suggested by asphericity does not extend to the smooth category in dimension 4. In particular, some closed aspherical topological 4-manifolds carry more than one smooth structure, up to diffeomorphism.

The word "closed" is important here. Many exotic phenomena in dimension 4 were already known for non-closed manifolds, but Theorem 1 produces exotic smooth structures in the compact, boundaryless setting. Moreover, the construction in (Davis et al. 2026) yields infinitely many such exotic pairs.

Update 22.09.2026. The paper (Davis et al. 2026) has now been published. The reference is updated to the published version.

References

Davis, Michael, Kyle Hayden, Jingyin Huang, Daniel Ruberman, and Nathan Sunukjian. 2026. "Exotic Aspherical 4-Manifolds." *Ann. Of Math. (2)* 204 (2): 777–94. https://doi.org/10.4007/annals.2026.204.2.6.