

# The existence of exotic aspherical 4-manifolds

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In a recent paper accepted for publication in the *Annals of Mathematics*, Davis, Hayden, Huang, Ruberman, and Sunukjian proved the existence of exotic pairs of smooth, closed, aspherical 4-manifolds (Davis et al. 2024).

Let  $X$  and  $Y$  be topological spaces. Two continuous maps  $F, G : X \rightarrow Y$  are said to be *homotopic* if there exists a continuous map

$$H : X \times [0, 1] \rightarrow Y$$

such that  $H(x, 0) = F(x)$  and  $H(x, 1) = G(x)$  for every  $x \in X$ . A continuous map  $F : X \rightarrow Y$  is a *homotopy equivalence* if there is a continuous map  $G : Y \rightarrow X$  such that  $G \circ F$  is homotopic to the identity map on  $X$ , and  $F \circ G$  is homotopic to the identity map on  $Y$ . In this case,  $G$  is called a homotopy inverse of  $F$ .

A path-connected topological  $k$ -manifold  $M$  is called *aspherical* if all of its higher homotopy groups vanish. This means that for every integer  $i \geq 2$ , every continuous map

$$\mathbb{S}^i \rightarrow M$$

from the  $i$ -sphere to  $M$  is homotopic to a constant map.

A topological manifold can sometimes support more than one smooth structure. We shall call two smooth manifolds  $M$  and  $N$  an *exotic pair* if they are homeomorphic but not diffeomorphic. Equivalently, after identifying  $M$  and  $N$  by a homeomorphism, one obtains two distinct smooth structures on the same underlying topological manifold, distinct up to diffeomorphism.

The *Borel conjecture* predicts that closed aspherical topological manifolds are topologically rigid: if  $M$  and  $N$  are closed aspherical topological manifolds and  $f : M \rightarrow N$  is a homotopy equivalence, then  $f$  should be homotopic to a homeomorphism. A natural smooth analogue asks whether two closed smooth aspherical  $n$ -manifolds that are homotopy equivalent must be diffeomorphic.

This smooth analogue is classical in dimensions  $n \leq 2$ . In dimension 3, it is known for orientable manifolds as a consequence of Perelman's proof of geometrization. In contrast, counterexamples had long been known in every dimension  $n \geq 5$ . Thus dimension 4 was the remaining case.

Davis, Hayden, Huang, Ruberman, and Sunukjian resolved this case by proving the following theorem.

**Theorem 1** *There exist smooth, closed, aspherical 4-manifolds that are homeomorphic but not diffeomorphic.*

Theorem 1 shows that the topological rigidity suggested by asphericity does not extend to the smooth category in dimension 4. In particular, some closed aspherical topological 4-manifolds carry more than one smooth structure, up to diffeomorphism.

The word “closed” is important here. Many exotic phenomena in dimension 4 were already known for non-closed manifolds, but Theorem 1 produces exotic smooth structures in the compact, boundaryless setting. Moreover, the construction in (Davis et al. 2024) yields infinitely many such exotic pairs.

## References

Davis, Michael, Kyle Hayden, Jingyin Huang, Daniel Ruberman, and Nathan Sunukjian. 2024. “Exotic Aspherical 4-Manifolds.” *arXiv Preprint arXiv:2411.19400*.