The existence of atoroidal surface bundles

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In a paper (Kent and Leininger 2024) recently accepted for publication in the *Annals of Mathematics*, Kent and Leininger constructed the first examples of compact atoroidal surface bundles over surfaces.

One natural way to build higher-dimensional manifolds from surfaces is to let one surface vary continuously over another. Let S and B be closed surfaces. A *surface bundle over B with fiber S* is a surjective continuous map

$$p : E \to B,$$

where E is a topological 4-manifold, such that every point $b \in B$ has an open neighbourhood $U \subset B$ for which there is a homeomorphism

$$\Phi : p^{-1}(U) \to U \times S$$

satisfying

$$p = \mathrm{pr}_1 \circ \Phi$$

on $p^{-1}(U)$. Here $U \times S$ has the product topology, and

$$\mathrm{pr}_1 : U \times S \to U$$

is the projection $\mathrm{pr}_1(u, s) = u$. Thus, although the total space E may be globally complicated, it looks locally like a product of an open set in B with the surface S.

For $b \in B$, the set

$$p^{-1}(b) = \{x \in E : p(x) = b\}$$

is called the *fiber* over b, and every fiber is homeomorphic to S. The manifold E is called the *total space*, and B is called the *base*. Such a bundle is often denoted by

$$S \to E \xrightarrow{p} B.$$

The simplest example is the product bundle $E = S \times B$, with projection

$$p : S \times B \to B, \qquad p(s, b) = b.$$

Let $\mathbb{Z}^2$ denote the group $\mathbb{Z} \times \mathbb{Z}$ with componentwise addition. If M is a path-connected manifold, let $\pi_1(M)$ denote its fundamental group. Since the fundamental group of the torus is isomorphic to $\mathbb{Z}^2$, the presence of a subgroup isomorphic to $\mathbb{Z}^2$ in $\pi_1(M)$ may be viewed as an algebraic trace of a torus. In the present context, a manifold M is called *atoroidal* if its fundamental group contains no subgroup isomorphic to $\mathbb{Z}^2$.

Many familiar surface bundles over surfaces are not atoroidal. For instance, if both S and B have positive genus, then the product bundle $S \times B$ has fundamental group

$$\pi_1(S \times B) \cong \pi_1(S) \times \pi_1(B),$$

which contains a copy of $\mathbb{Z}^2$. Thus it is natural, but far from obvious, to ask whether atoroidal surface bundles

$$S \to E \to B$$

exist in the genuinely surface-by-surface case, where both S and B are closed orientable surfaces of genus at least 2.

Kent and Leininger answered this question affirmatively in their paper (Kent and Leininger 2024), constructing the first such examples.

Theorem 1 *There exists a closed aspherical* 4*-manifold* E *that is the total space of a surface bundle*

$$S \to E \xrightarrow{p} B,$$

where S *and* B *are closed orientable surfaces of genus at least* 2*, and whose fundamental group* $\pi_1(E)$ *contains no subgroup isomorphic to* $\mathbb{Z}^2$*.*

Theorem 1 shows that surface bundles over surfaces can have much subtler global topology than product bundles. Locally, the total space still looks like a product of two surfaces, but globally its fundamental group avoids even the basic rank-two abelian subgroup $\mathbb{Z}^2$. In this sense, Kent and Leininger's examples demonstrate that the local product structure of a surface bundle imposes far fewer restrictions on the large-scale algebraic topology of the total space than one might first expect.

References

Kent, Autumn E, and Christopher J Leininger. 2024. "Atoroidal Surface Bundles." *arXiv Preprint arXiv:2405.12067*.