Ben Green’s talk on the polynomial Freiman-Ruzsa conjecture

Bogdan Grechuk • 20 Apr 2026

This post continues our series of online seminars devoted to accessible presentations of some of the most significant mathematical theorems of the 21st century. I am delighted to report that, on April 15, Prof. Harald Helfgott delivered an accessible lecture on growth, expansion, and escape in groups. If you missed the talk, the recording is now available on YouTube.

The next lecture in the series will take place on

$$\textbf{Wednesday, May 20, 2026, at 4:00 PM (UK time).}$$

Prof. Ben Green (University of Oxford, UK) will speak on the resolution of *the polynomial Freiman–Ruzsa conjecture*.

To join the talk, simply click here at the scheduled time; no registration is required. For a full list of upcoming seminars, please visit https://th21.le.ac.uk/next-talks/.

For announcements of other talks, as well as descriptions of recent major mathematical breakthroughs, see the full list of posts on this blog.

Below is my description of the topic of the talk. This is **not** the abstract received from the speaker.

For sets of integers A and B, write

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

If A is an arithmetic progression, then $|A + A| < 2|A|$. A deep theorem of Freiman from 1973 (Freiman 1973) describes all possible examples of sets A such that $|A + A| \leq k|A|$ for fixed k. An important line of research is to obtain analogues of Freiman’s theorem for groups other than $\mathbb{Z}$.

Let A be a subset of a group G with operation $+$. In which cases can one have $|A + A| \leq k|A|$ for fixed k? An obvious example is when $A = H$ is a finite subgroup of G, in which case $|A + A| = |A|$. The same equality holds if A is a coset of a finite subgroup H, that is,

$$A = \{g + h : h \in H\}$$

for some $g \in G$. More generally, if A is a subset of a subgroup H such that $|H| \leq k|A|$, then $A + A \subseteq H$ implies $|A + A| \leq |H| \leq k|A|$. Similarly, if A can be covered by m cosets of H, then $A + A$ can be covered by at most m^2 cosets, so

$$|A + A| \leq m^2|H|.$$

Thus this quantity is bounded above by $k|A|$ whenever $|H| \leq k|A|/m^2$. A natural question is whether one can reverse this reasoning: starting from the assumption $|A + A| \leq k|A|$, can one conclude that A is contained in a subgroup H of size at most $f(k)|A|$, or at least that A can be covered by $g(k)$ cosets of such a subgroup?

In 1999, Ruzsa (Ruzsa 1999) proved that this is indeed the case for certain groups. More precisely, let $r \geq 2$ be an integer, let G be an abelian group in which every element has order at most r, and let $A \subset G$ be a finite set such that $|A + A| \leq k|A|$. Then A is contained in a subgroup H of G satisfying

$$|H| \leq f(r, k)|A|,$$

where $f(r, k) = k^2 r^{k^4}$. Ruzsa also cited a conjecture, which he attributed to Marton, asserting that under the same assumptions there should exist a subgroup H of G such that $|H| \leq |A|$ and A is contained in the union of at most $2k^c$ cosets of H, where the constant c depends only on r. This statement became widely known as the *polynomial Freiman–Ruzsa conjecture*, and it has been described as the “holy grail of additive combinatorics.”

An important family of groups to which the conjecture applies is $G = F_2^n$, the group of binary strings of length n with componentwise addition modulo 2. This is an abelian group in which every element has order 2. In 2025, Gowers, Green, Manners, and Tao (Gowers et al. 2025) proved the polynomial Freiman–Ruzsa conjecture for this family of groups.

Theorem 1 *Suppose that $A \subset F_2^n$ satisfies $|A + A| \leq k|A|$. Then A can be covered by at most $2k^c$ cosets of some subgroup H of F_2^n of size at most $|A|$. In fact, one may take $c = 12$.*

Liao (Liao 2024) later observed that Theorem 1 remains true with $c = 9$. In the opposite direction, it is known (Green and Tao 2009) that one must have $c > 1.466$.

Theorem 1 has many corollaries and equivalent formulations. In particular, it was observed in (Green et al. 2025) that it implies the so-called “weak polynomial Freiman–Ruzsa conjecture over $\mathbb{Z}$”. The latter predicts that if $d \geq 1$

is an integer and A is a finite subset of $\mathbb{Z}^d$ satisfying $|A + A| \leq k|A|$, then there exists a subset $A' \subseteq A$ with $|A'| \geq k^{-C_1}|A|$ whose dimension is at most $C_2 \log_2 k$. Theorem 1 implies that this is true, in fact with $C_1 = 17$ and $C_2 = 40$.

In later work (Gowers et al. 2024), the same authors generalized Theorem 1 to arbitrary abelian groups of bounded torsion, thereby establishing the polynomial Freiman–Ruzsa conjecture in full generality in that setting. Recall that an abelian group $G = (G, +)$ has torsion m if $mg = 0$ for all $g \in G$. The main result of (Gowers et al. 2024) states that if G is an abelian group of torsion $m \geq 2$ and $A \subseteq G$ is a finite non-empty set satisfying $|A + A| \leq k|A|$, then A can be covered by at most $(2k)^{O(m^3)}$ cosets of some subgroup H of G of size at most $|A|$. A canonical family of examples is given by $G = F_p^n$ for a fixed prime p, where every such group has torsion $m = p$.

For further progress toward a version of the polynomial Freiman–Ruzsa conjecture for arbitrary abelian groups, with no restriction on torsion, see (Raghavan 2025).

References

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