# On the density of primitive sets of large integers

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It is no longer surprising that modern AI models can solve open problems in mathematics, nor even that they can settle some of the questions posed by Paul Erdős. I am surprised, however, that ChatGPT has now resolved a problem still described as open in the forthcoming second edition of my book *Landscape of 21st Century Mathematics*. The book discusses *major* theorems proved in the 21st century and *major* problems that remain unsolved. It is therefore truly remarkable that ChatGPT was able to settle an open problem of this level.

The problem concerns primitive sets of integers. A set $\mathcal{A}$ of integers greater than $1$ is called *primitive* if no element of $\mathcal{A}$ divides another. An obvious example of an infinite primitive set is the set $\mathcal{P}$ of prime numbers. More generally, for any $k \geq 1$, the set $\mathcal{P}_k$ of integers with exactly $k$ prime factors, counted with multiplicity, is primitive. Although the upper asymptotic density of a primitive set can be made arbitrarily close to $1/2$ (Besicovitch 1935), Erdős (Erdős 1935) proved that the lower asymptotic density of every primitive set $\mathcal{A}$ is $0$, and moreover that

$$f(\mathcal{A}) := \sum_{a \in \mathcal{A}} \frac{1}{a \log a} \leq C \tag{1}$$

for some finite constant $C$ independent of $\mathcal{A}$. This naturally raises the question of the smallest possible such constant. For the set of primes one has $f(\mathcal{P}) = 1.6366\ldots$, and Erdős conjectured that $(1)$ holds with $C = f(\mathcal{P})$; in other words, that $f(\mathcal{A}) \leq f(\mathcal{P})$ for every primitive set $\mathcal{A}$. This became known as the Erdős primitive set conjecture and it remained open for several decades. In 1993, Erdős and Zhang (Erdős and Zhang 1993) established $(1)$ with $C = 1.84$. In 2019, Lichtman and Pomerance (Lichtman and Pomerance 2019) improved this bound to $C \approx 1.781$.

A particularly appealing strategy for proving the conjecture is to partition the integers in a primitive set $\mathcal{A}$ according to their least prime factor. Thus, let $\mathcal{A}_2$ denote the even elements of $\mathcal{A}$, let $\mathcal{A}_3$ denote the odd elements divisible by $3$, and more generally let $\mathcal{A}_p$ be the set of all integers in $\mathcal{A}$ whose smallest prime factor is $p$. The Erdős primitive set conjecture would follow immediately if one could show that

$$f(\mathcal{A}_p) \leq \frac{1}{p \log p}$$

for every prime $p$. Before 2023, this inequality was known for all odd primes up to $10^8$. In 2023, Lichtman (Lichtman 2023) proved it for all odd primes. The case $p = 2$, however, remained open. Nevertheless, Lichtman (Lichtman 2023) was able to circumvent this obstacle and establish the Erdős primitive set conjecture in full.

**Theorem 1** *For any primitive set $\mathcal{A}$, we have $f(\mathcal{A}) \leq f(\mathcal{P})$.*

Theorem 1 shows that the primes are the "densest" primitive set when density is measured by the function $f$. One can also ask for the densest primitive sets among large integers: namely, what is the largest possible value of $f(\mathcal{A})$ for primitive sets consisting only of integers exceeding $x$, as $x \to \infty$? It is known that for the sets $\mathcal{P}_k$ of integers with exactly $k$ prime factors,

$$\lim_{k \to \infty} f(\mathcal{P}_k) = 1,$$

and it was conjectured that these sets are asymptotically the densest possible, in the sense that

$$\lim_{x \to \infty} \sup_{\mathcal{A} \in \mathcal{A}_x} f(\mathcal{A}) = 1, \tag{2}$$

where $\mathcal{A}_x$ denotes the class of all primitive subsets of $[x, \infty)$. Lichtman (Lichtman 2023) also made progress on this problem by proving that the limit in $(2)$ is less than 1.4. Both Theorem 1 and conjecture $(2)$ are discussed in the forthcoming second edition of my book among the major developments and open problems of contemporary mathematics.

In 2026, this conjecture was fully confirmed automatically by GPT-5.4 Pro, which proved the following explicit result.

**Theorem 2** *There exists a constant $c$ such that $f(\mathcal{A}) \leq 1 + c(\log x)^{-1}$ for all $x > 1$ and all $\mathcal{A} \in \mathcal{A}_x$.*

This result is striking not only because it settles a natural conjecture in the theory of primitive sets, but also because it illustrates how rapidly AI-assisted mathematics is advancing. Problems that were, until very recently, part of the landscape of future research are now beginning to yield to automated reasoning, suggesting that the boundary between human and machine mathematical discovery is changing in a fundamental way.

# References

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