The proof of Catalan’s conjecture

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see this post.

My choice for 2002 in Number Theory is Mihailescu’s resolution of the famous Catalan conjecture, which asserts that the integers $8 = 2^3$ and $9 = 3^2$ are the only consecutive positive integers that are both perfect powers. This result was proved in 2002 and published in 2004 (Mihailescu 2004).

A large part of number theory is concerned with Diophantine equations: polynomial equations whose solutions are required to be integers. The most basic question is whether such equations have any integer solutions at all. When some obvious solutions are already known, the next natural question is whether there are any others.

Catalan (Catalan 1844) conjectured in 1844 that $8 = 2^3$ and $9 = 3^2$ are the only consecutive positive integers that are perfect powers. Equivalently, he predicted that $3^2 - 2^3 = 1$ is the only solution in positive integers to

$$x^a - y^b = 1 \tag{1}$$

with $x, y > 0$ and $a, b > 1$.

A simple reduction shows that it is enough to consider the case in which a and b are prime. Indeed, if x, y, a, b satisfy (1), and if p and q are prime divisors of a and b respectively, then

$$\left(x^{a/p}\right)^p - \left(y^{b/q}\right)^q = 1.$$

Thus any solution gives rise to one in which the exponents are prime. In 1850, Lebesgue (Lebesgue 1850) proved that there are no positive integers x, y and $a > 1$ such that $x^a = y^2 + 1$, settling the case $b = 2$ of (1). In 1960, Ko Chao (Chao 1960) resolved the case $a = 2$. After these results, the conjecture was reduced to the case in which a and b are distinct odd primes.

A major breakthrough came in 1976, when Tijdeman (Tijdeman 1976) proved that if (1) has any solution in positive integers other than $3^2 - 2^3 = 1$, then it must have one with $a, b < C$, where C is an absolute constant. Later that same year, Langevin (Langevin 1976) showed that one may take $C = 10^{110}$. Since for any fixed pair (a, b) the equation (1) can be solved in finite time, Tijdeman's theorem turned Catalan's conjecture into a finite computation in principle, although still one far beyond practical reach.

Mihailescu's decisive advance was to uncover a remarkably strong arithmetic constraint on any hypothetical solution. In 2003 (Mihailescu 2003) he proved that if a, b are prime exponents arising from a solution to (1), then

$$a^{b-1} \equiv 1 \pmod{b^2} \qquad \text{and} \qquad b^{a-1} \equiv 1 \pmod{a^2}.$$

A pair of primes satisfying these congruences is called a *double Wieferich prime pair*. This necessary condition is extraordinarily restrictive, and it sharply narrows the landscape in which a counterexample could possibly live. Mihailescu then completed the argument in 2004 (Mihailescu 2004), proving Catalan's conjecture in full.

Theorem 1 *The integers $8 = 2^3$ and $9 = 3^2$ are the only consecutive positive integers which are perfect powers.*

What makes this theorem especially appealing to me is the contrast between the simplicity of its statement and the depth of its proof. The question can be explained to anyone in a sentence, yet its resolution required delicate ideas from algebraic and Diophantine number theory. That combination is part of what makes Catalan's conjecture so memorable: it is an elementary-looking problem whose solution reveals unexpectedly rich structure. Mihailescu's theorem did more than settle a conjecture from 1844; it brought a long line of work on exponential Diophantine equations to a striking and satisfying conclusion.

References

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