The ABC sum-product problem

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In a paper (Orponen and Shmerkin 2026), recently accepted for publication in the *Journal of the American Mathematical Society*, Orponen and Shmerkin gave a positive solution to the ABC sum-product problem.

Sum-product theorems over the integers, discussed in this previous blog post, assert that for a finite set $A \subset \mathbb{Z}$, the sumset $A + A$ and the product set $A \cdot A$ cannot both be small. In that setting, the relevant sets are finite, and their size is measured by cardinality. In 2003, Bourgain (Bourgain 2003) established an analogue for infinite subsets of the real line, where "size" is quantified using δ-covering numbers. Let

$$B(x, r) = \{y \in \mathbb{R} : |x - y| \leq r\}$$

denote the ball in $\mathbb{R}$ centred at $x \in \mathbb{R}$ with radius $r > 0$. For $\delta > 0$ and any set $X \subset \mathbb{R}$, write $|X|_\delta$ for the minimum number of balls of radius δ needed to cover X. Bourgain proved that for any $s, t \in (0, 1)$, there exists $\epsilon = \epsilon(s, t) > 0$ such that, for all sufficiently small $\delta > 0$, the following holds: if $A \subseteq [1, 2]$ satisfies $|A|_\delta \leq \delta^{-t}$ and

$$|A \cap B(x, r)|_\delta \leq \delta^{-\epsilon} \cdot r^s |A|_\delta \quad \text{for all } x \in [1, 2] \text{ and } r \in [\delta, 1],$$

then

$$\max\{|A + A|_\delta,\ |A \cdot A|_\delta\} \geq |A|_\delta^{1+\epsilon}.$$

This theorem has several striking consequences. For instance, it implies that if $A, B, C \subset \mathbb{R}$ are Borel sets with $\dim(A) = \dim(B) \in (0, 1)$ and $\dim(C) > 0$, then there exists $c \in C$ such that

$$\dim(A + cB) > \dim(A). \tag{1}$$

In 2022, Orponen (Orponen 2022) conjectured that (1) should remain valid in the more general regime where $0 \leq \dim(B) \leq \dim(A) < 1$ and $\dim(C) > \dim(A) - \dim(B)$. He also showed that, if true, this statement would be sharp. This became known as the *ABC sum-product problem*. In 2026, Orponen and Shmerkin (Orponen and Shmerkin 2026) confirmed the conjecture.

Theorem 1 *Let* $0 < \beta \leq \alpha < 1$ *and* $\kappa > 0$. *Then there exists* $\eta = \eta(\alpha, \beta, \kappa) > 0$ *such that if* $A, B, C \subset \mathbb{R}$ *are Borel sets with* $\dim(A) = \alpha$, $\dim(B) = \beta$, *and* $\dim(C) \geq \alpha - \beta + \kappa$, *then there exists* $c \in C$ *such that*

$$\dim(A + cB) \geq \dim(A) + \eta.$$

In other words, the theorem shows that the threshold predicted by Orponen is indeed the correct one: as soon as C has dimension exceeding $\dim(A) - \dim(B)$ by a positive amount, one can find a parameter $c \in C$ for which the set $A + cB$ has strictly larger dimension than A. This gives a definitive resolution of the ABC sum-product problem and marks a major advance in the theory of sum-product phenomena over real numbers.

Update 13.04.2026 The paper (Orponen and Shmerkin 2026) is now published. The reference is updated to the published version.

References

Bourgain, J. 2003. "On the Erdős-Volkmann and Katz-Tao Ring Conjectures." *Geom. Funct. Anal.* 13 (2): 334–65. https://doi.org/10.1007/s000390300008.

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