# Affine Non-Concentration and Fourier Decay

Bogdan Grechuk • 6 Apr 2026

In a paper (Khalil 2026), recently accepted for publication in the *Journal of the American Mathematical Society*, Osama Khalil proved that the Fourier transform of affinely non-concentrated measures enjoys polynomial decay outside of a sparse set. This fits into a broad principle in harmonic analysis: when a measure is sufficiently spread out in physical space and does not cluster along lower-dimensional structures, its Fourier transform should exhibit cancellation and therefore become small at high frequencies. Khalil's theorem gives a precise version of this principle for measures on $\mathbb{R}^d$.

The term "transform" is often used to name a function $h : X \to Y$ when the domain $X$ is itself a set of functions, because $h$ "transforms" functions into functions. One of the most prominent examples, with countless applications in analysis, partial differential equations, number theory, and signal processing, is the *Fourier transform*

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i t x}\mathrm{d}x, \quad t \in \mathbb{R}, \tag{1}$$

of an integrable function $f : \mathbb{R} \to \mathbb{C}$.

Very roughly, the Fourier transform measures how much oscillation of frequency $t$ is present in the function $f$. Large values of $\hat{f}(t)$ indicate that the oscillation $\mathrm{e}^{2\pi i t x}$ correlates strongly with $f$, while decay of $\hat{f}(t)$ as $|t| \to \infty$ reflects cancellation coming from oscillation. This makes the Fourier transform a powerful bridge between geometric or spatial structure and frequency-space behavior.

The Fourier transform can also be applied to measures. Recall that a Borel probability measure $\mu$ on $\mathbb{R}^d$ is a rule that assigns a mass $\mu(E) \in [0,1]$ to each Borel subset $E \subseteq \mathbb{R}^d$, with total mass $\mu(\mathbb{R}^d) = 1$. This viewpoint is useful because measures allow us to describe distributions of mass that may be singular, fractal, or otherwise not given by a density function.

For a measure $\mu$ on $\mathbb{R}^d$, its Fourier transform is the function

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x\rangle}\, d\mu(x), \qquad \xi \in \mathbb{R}^d,$$

where $\xi \in \mathbb{R}^d$ is the *frequency variable*, $x \in \mathbb{R}^d$ is the original *spatial variable*, $\langle \xi, x \rangle$ is the usual dot product, and $i = \sqrt{-1}$. As in the classical case, decay of $\widehat{\mu}(\xi)$ for large $\|\xi\|$ reflects oscillatory cancellation. If the measure is highly concentrated on simple geometric objects, such cancellation can fail; if the measure is genuinely spread through many directions, one expects stronger decay.

In the recently accepted paper, Khalil (Khalil 2026) proved that the Fourier transform of probability measures on $\mathbb{R}^d$ that do not concentrate near proper affine hyperplanes enjoys polynomial decay outside of a sparse set of frequencies. The geometric condition rules out the possibility that, at small scales, most of the mass sits near a lower-dimensional flat set. Intuitively, it says that no matter where one looks and at what scale, the measure retains genuinely $d$-dimensional behavior rather than collapsing toward a hyperplane.

Formally, $\mu$ is called *uniformly affinely non-concentrated* if for every $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every $x$ in the support of $\mu$, every radius $0 < r \leq 1$, and every affine hyperplane $W \subset \mathbb{R}^d$,

$$\mu\big(W(\varepsilon r) \cap B(x,r)\big) \leq \delta(\varepsilon)\, \mu(B(x,r)),$$

where $B(x,r)$ is the ball of radius $r$ centered at $x$, and $W(\varepsilon r)$ is the $\varepsilon r$-neighborhood of the hyperplane $W$, consisting of all points $y \in \mathbb{R}^d$ whose distance to $W$ is at most $\varepsilon r$.

This condition is scale-invariant in spirit: inside any ball $B(x,r)$, only a small proportion of the mass is allowed to lie very close to any affine hyperplane. The smaller the relative thickness parameter $\varepsilon$ is, the smaller that proportion must become. In other words, the measure cannot repeatedly pile up near codimension-one affine structures.

Now we are ready to state Khalil's theorem.

**Theorem 1** *Let $\mu$ be a compactly supported Borel probability measure on $\mathbb{R}^d$, that is uniformly affinely non-concentrated. Let $\widehat{\mu}$ be its Fourier transform. Then for every $\varepsilon > 0$, there exists $\tau > 0$ such that for all $T \geq 1$,*

$$\left|\{\xi \in \mathbb{R}^d : \ \|\xi\| \leq T, \ |\widehat{\mu}(\xi)| > T^{-\tau}\}\right| \leq C_{\varepsilon,\mu} T^{\varepsilon}.$$

Here, $\|\cdot\|$ denotes the usual Euclidean norm, and $|\cdot|$ is Lebesgue measure (ordinary volume) in frequency space. Thus, Theorem 1 says that inside the ball $\{\|\xi\| \leq T\}$, the set of frequencies where $\widehat{\mu}(\xi)$ remains noticeably large occupies very little volume. Since $T^{\varepsilon}$ grows much more slowly than the ambient volume scale $T^d$, this exceptional set is sparse in a very strong sense.

It is worth emphasizing what the theorem does and does not say. It does not assert pointwise polynomial decay for every frequency $\xi$; instead, it shows that the frequencies where decay fails form a set of very small measure. This is often exactly the right kind of statement for applications, since many arguments in analysis, dynamics, and ergodic theory only need control away from a sufficiently small exceptional set.

The result is in agreement with a classical theme in Fourier analysis: spread-out geometric structure in physical space implies decay in frequency space. Results of this type are extremely useful. In fact, Theorem 1 was not the main result of (Khalil 2026), but rather a key tool used to prove some deep theorems in ergodic theory and dynamical systems.

**Update 13.04.2026** The paper (Khalil 2026) is now published. The reference is updated to the published version.

# References

Khalil, Osama. 2026. “Exponential Mixing via Additive Combinatorics.” *J. Amer. Math. Soc.* 39 (3): 781–856. https://doi.org/10.1090/jams/1076.