

# Affine Non-Concentration and Fourier Decay

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In a paper (Khalil 2023), recently accepted for publication in the *Journal of the American Mathematical Society*, Osama Khalil proved that the Fourier transform of affinely non-concentrated measures enjoys polynomial decay outside of a sparse set. This fits into a broad principle in harmonic analysis: when a measure is sufficiently spread out in physical space and does not cluster along lower-dimensional structures, its Fourier transform should exhibit cancellation and therefore become small at high frequencies. Khalil's theorem gives a precise version of this principle for measures on  $\mathbb{R}^d$ .

The term “transform” is often used to name a function  $h : X \rightarrow Y$  when the domain  $X$  is itself a set of functions, because  $h$  “transforms” functions into functions. One of the most prominent examples, with countless applications in analysis, partial differential equations, number theory, and signal processing, is the *Fourier transform*

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx} dx, \quad t \in \mathbb{R}, \quad (1)$$

of an integrable function  $f : \mathbb{R} \rightarrow \mathbb{C}$ .

Very roughly, the Fourier transform measures how much oscillation of frequency  $t$  is present in the function  $f$ . Large values of  $\hat{f}(t)$  indicate that the oscillation  $e^{2\pi itx}$  correlates strongly with  $f$ , while decay of  $\hat{f}(t)$  as  $|t| \rightarrow \infty$  reflects cancellation coming from oscillation. This makes the Fourier transform a powerful bridge between geometric or spatial structure and frequency-space behavior.

The Fourier transform can also be applied to measures. Recall that a Borel probability measure  $\mu$  on  $\mathbb{R}^d$  is a rule that assigns a mass  $\mu(E) \in [0, 1]$  to each Borel subset  $E \subseteq \mathbb{R}^d$ , with total mass  $\mu(\mathbb{R}^d) = 1$ . This viewpoint is useful because measures allow us to describe distributions of mass that may be singular, fractal, or otherwise not given by a density function.

For a measure  $\mu$  on  $\mathbb{R}^d$ , its Fourier transform is the function

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle} d\mu(x), \quad \xi \in \mathbb{R}^d,$$

where  $\xi \in \mathbb{R}^d$  is the *frequency variable*,  $x \in \mathbb{R}^d$  is the original *spatial variable*,  $\langle \xi, x \rangle$  is the usual dot product, and  $i = \sqrt{-1}$ . As in the classical case, decay of  $\widehat{\mu}(\xi)$  for large  $\|\xi\|$  reflects oscillatory cancellation. If the measure is highly concentrated on simple geometric objects, such cancellation can fail; if the measure is genuinely spread through many directions, one expects stronger decay.

In the recently accepted paper, Khalil (Khalil 2023) proved that the Fourier transform of probability measures on  $\mathbb{R}^d$  that do not concentrate near proper affine hyperplanes enjoys polynomial decay outside of a sparse set of frequencies. The geometric condition rules out the possibility that, at small scales, most of the mass sits near a lower-dimensional flat set. Intuitively, it says that no matter where one looks and at what scale, the measure retains genuinely  $d$ -dimensional behavior rather than collapsing toward a hyperplane.

Formally,  $\mu$  is called *uniformly affinely non-concentrated* if for every  $\varepsilon > 0$ , there is a number  $\delta(\varepsilon) > 0$  with  $\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , such that for every  $x$  in the support of  $\mu$ , every radius  $0 < r \leq 1$ , and every affine hyperplane  $W \subset \mathbb{R}^d$ ,

$$\mu(W(\varepsilon r) \cap B(x, r)) \leq \delta(\varepsilon) \mu(B(x, r)),$$

where  $B(x, r)$  is the ball of radius  $r$  centered at  $x$ , and  $W(\varepsilon r)$  is the  $\varepsilon r$ -neighborhood of the hyperplane  $W$ , consisting of all points  $y \in \mathbb{R}^d$  whose distance to  $W$  is at most  $\varepsilon r$ .

This condition is scale-invariant in spirit: inside any ball  $B(x, r)$ , only a small proportion of the mass is allowed to lie very close to any affine hyperplane. The smaller the relative thickness parameter  $\varepsilon$  is, the smaller that proportion must become. In other words, the measure cannot repeatedly pile up near codimension-one affine structures.

Now we are ready to state Khalil's theorem.

**Theorem 1** *Let  $\mu$  be a compactly supported Borel probability measure on  $\mathbb{R}^d$ , that is uniformly affinely non-concentrated. Let  $\widehat{\mu}$  be its Fourier transform. Then for every  $\varepsilon > 0$ , there exists  $\tau > 0$  such that for all  $T \geq 1$ ,*

$$|\{\xi \in \mathbb{R}^d : \|\xi\| \leq T, |\widehat{\mu}(\xi)| > T^{-\tau}\}| \leq C_{\varepsilon, \mu} T^\varepsilon.$$

Here,  $\|\cdot\|$  denotes the usual Euclidean norm, and  $|\cdot|$  is Lebesgue measure (ordinary volume) in frequency space. Thus, Theorem 1 says that inside the ball  $\{\|\xi\| \leq T\}$ , the set of frequencies where  $\widehat{\mu}(\xi)$  remains noticeably large occupies very little volume. Since  $T^\varepsilon$  grows much more slowly than the ambient volume scale  $T^d$ , this exceptional set is sparse in a very strong sense.

It is worth emphasizing what the theorem does and does not say. It does not assert pointwise polynomial decay for every frequency  $\xi$ ; instead, it shows that the frequencies where decay fails form a set of very small measure. This is often exactly the right kind of statement for applications, since many arguments in analysis, dynamics, and ergodic theory only need control away from a sufficiently small exceptional set.

The result is in agreement with a classical theme in Fourier analysis: spread-out geometric structure in physical space implies decay in frequency space. Results of this type are extremely useful. In fact, Theorem 1 was not the main result of (Khalil 2023), but rather a key tool used to prove deeper theorems in ergodic theory and dynamical systems.

## References

Khalil, Osama. 2023. “Exponential Mixing via Additive Combinatorics.” *arXiv Preprint arXiv:2305.00527*.