Affine Non-Concentration and Fourier Decay

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In a paper (Khalil 2026), recently accepted for publication in the *Journal of the American Mathematical Society*, Osama Khalil proved that the Fourier transform of affinely non-concentrated measures enjoys polynomial decay outside of a sparse set. This fits into a broad principle in harmonic analysis: when a measure is sufficiently spread out in physical space and does not cluster along lower-dimensional structures, its Fourier transform should exhibit cancellation and therefore become small at high frequencies. Khalil's theorem gives a precise version of this principle for measures on $\mathbb{R}^d$.

The term "transform" is often used to name a function $h : X \to Y$ when the domain X is itself a set of functions, because h "transforms" functions into functions. One of the most prominent examples, with countless applications in analysis, partial differential equations, number theory, and signal processing, is the *Fourier transform*

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi itx}\mathrm{d}x, \quad t \in \mathbb{R}, \tag{1}$$

of an integrable function $f : \mathbb{R} \to \mathbb{C}$.

Very roughly, the Fourier transform measures how much oscillation of frequency t is present in the function f. Large values of $\hat{f}(t)$ indicate that the oscillation $\mathrm{e}^{2\pi itx}$ correlates strongly with f, while decay of $\hat{f}(t)$ as $|t| \to \infty$ reflects cancellation coming from oscillation. This makes the Fourier transform a powerful bridge between geometric or spatial structure and frequency-space behavior.

The Fourier transform can also be applied to measures. Recall that a Borel probability measure μ on $\mathbb{R}^d$ is a rule that assigns a mass $\mu(E) \in [0, 1]$ to each Borel subset $E \subseteq \mathbb{R}^d$, with total mass $\mu(\mathbb{R}^d) = 1$. This viewpoint is useful because measures allow us to describe distributions of mass that may be singular, fractal, or otherwise not given by a density function.

For a measure μ on $\mathbb{R}^d$, its Fourier transform is the function

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\langle \xi, x\rangle}\, d\mu(x), \qquad \xi \in \mathbb{R}^d,$$

where $\xi \in \mathbb{R}^d$ is the *frequency variable*, $x \in \mathbb{R}^d$ is the original *spatial variable*, $\langle \xi, x \rangle$ is the usual dot product, and $i = \sqrt{-1}$. As in the classical case, decay of $\widehat{\mu}(\xi)$ for large $\|\xi\|$ reflects oscillatory cancellation. If the measure is highly concentrated on simple geometric objects, such cancellation can fail; if the measure is genuinely spread through many directions, one expects stronger decay.

In the recently accepted paper, Khalil (Khalil 2026) proved that the Fourier transform of probability measures on $\mathbb{R}^d$ that do not concentrate near proper affine hyperplanes enjoys polynomial decay outside of a sparse set of frequencies. The geometric condition rules out the possibility that, at small scales, most of the mass sits near a lower-dimensional flat set. Intuitively, it says that no matter where one looks and at what scale, the measure retains genuinely d-dimensional behavior rather than collapsing toward a hyperplane.

Formally, μ is called *uniformly affinely non-concentrated* if for every $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every x in the support of μ, every radius $0 < r \leq 1$, and every affine hyperplane $W \subset \mathbb{R}^d$,

$$\mu\big(W(\varepsilon r) \cap B(x, r)\big) \leq \delta(\varepsilon)\, \mu(B(x, r)),$$

where $B(x, r)$ is the ball of radius r centered at x, and $W(\varepsilon r)$ is the εr-neighborhood of the hyperplane W, consisting of all points $y \in \mathbb{R}^d$ whose distance to W is at most εr.

This condition is scale-invariant in spirit: inside any ball $B(x, r)$, only a small proportion of the mass is allowed to lie very close to any affine hyperplane. The smaller the relative thickness parameter ε is, the smaller that proportion must become. In other words, the measure cannot repeatedly pile up near codimension-one affine structures.

Now we are ready to state Khalil's theorem.

Theorem 1 *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$, that is uniformly affinely non-concentrated. Let $\widehat{\mu}$ be its Fourier transform. Then for every $\varepsilon > 0$, there exists $\tau > 0$ such that for all $T \geq 1$,*

$$\Big|\{\xi \in \mathbb{R}^d : \ \|\xi\| \leq T, \ |\widehat{\mu}(\xi)| > T^{-\tau}\}\Big| \leq C_{\varepsilon,\mu} T^{\varepsilon}.$$

Here, $\|\cdot\|$ denotes the usual Euclidean norm, and $|\cdot|$ is Lebesgue measure (ordinary volume) in frequency space. Thus, Theorem 1 says that inside the ball $\{\|\xi\| \leq T\}$, the set of frequencies where $\widehat{\mu}(\xi)$ remains noticeably large occupies very little volume. Since T^{ε} grows much more slowly than the ambient volume scale T^d, this exceptional set is sparse in a very strong sense.

It is worth emphasizing what the theorem does and does not say. It does not assert pointwise polynomial decay for every frequency ξ; instead, it shows that the frequencies where decay fails form a set of very small measure. This is often exactly the right kind of statement for applications, since many arguments in analysis, dynamics, and ergodic theory only need control away from a sufficiently small exceptional set.

The result is in agreement with a classical theme in Fourier analysis: spread-out geometric structure in physical space implies decay in frequency space. Results of this type are extremely useful. In fact, Theorem 1 was not the main result of (Khalil 2026), but rather a key tool used to prove some deep theorems in ergodic theory and dynamical systems.

Update 13.04.2026 The paper (Khalil 2026) is now published. The reference is updated to the published version.

References

Khalil, Osama. 2026. “Exponential Mixing via Additive Combinatorics.” *J. Amer. Math. Soc.* 39 (3): 781–856. https://doi.org/10.1090/jams/1076.