The unknotting number is not additive

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In a paper (Brittenham and Hermiller 2025), recently accepted for publication in the *Annals of Mathematics*, Brittenham and Hermiller proved that the unknotting number is not additive under connected sum, disproving a long-standing conjecture in knot theory.

Recall from this previous blog post that a *knot* is a closed, non-self-intersecting curve embedded in $\mathbb{R}^3$. A *knot diagram* is a projection of a knot to the plane that is injective except at finitely many transverse double points, together with over/under information at each crossing. A diagram with no crossings is called *trivial*. The *Reidemeister moves* are the three local operations on knot diagrams consisting of twisting or untwisting a strand, sliding one strand over another, and sliding a strand across a crossing.

A *crossing switch* is the operation of selecting a crossing C and reversing its over/under information. It is easy to check that any diagram of a knot K can be transformed into the trivial diagram by a sequence of Reidemeister moves and crossing switches. The minimal number of crossing switches required in such a sequence is denoted $u(K)$ and is called the *unknotting number* of K. Since any two diagrams of the same knot can be transformed into one another using only Reidemeister moves, the unknotting number depends only on the knot K itself and not on the choice of diagram. Intuitively, $u(K)$ measures the minimum number of times the knot K must be passed through itself in order to become the unknot. By definition, $u(K) = 0$ if and only if K is the unknot, and more generally $u(K)$ gives a natural measure of how far K is from being trivial.

Any explicit sequence that unties a knot K using n crossing switches yields the upper bound $u(K) \leq n$. Proving a lower bound is much harder: to show that $u(K) > m$, one must prove that no sequence using at most m crossing switches can unknot K, regardless of how many Reidemeister moves are used or how complicated the intermediate diagrams become. For this reason, computing unknotting numbers is often highly nontrivial, and for many knots K the exact value of $u(K)$ remains unknown.

The *connected sum* of two knots K_1 and K_2, denoted $K_1 \# K_2$, is obtained by starting with disjoint diagrams of K_1 and K_2, deleting a short arc from each, and joining the resulting endpoints without introducing any new crossings. A knot is called *prime* if it cannot be expressed as a nontrivial connected sum. Every knot admits a unique decomposition into prime knots, up to order. It is immediate that

$$u(K_1 \# K_2) \leq u(K_1) + u(K_2).$$

A classical conjecture, already implicit in Wendt's 1937 work (Wendt 1937) and stated explicitly many times since, asserted that equality always holds. If true, this would reduce the problem of determining unknotting numbers of arbitrary knots to the corresponding problem for prime knots.

The conjecture is trivial when $u(K_1) = 0$ or $u(K_2) = 0$. In 1985, Scharlemann (Scharlemann 1985) proved it in the first genuinely nontrivial case, namely when $u(K_1) = u(K_2) = 1$. It has also been established under various additional hypotheses on K_1 and K_2. However, in their recently accepted Annals paper, Brittenham and Hermiller (Brittenham and Hermiller 2025) showed that the conjecture is false in general by constructing explicit counterexamples.

Theorem 1 *There exist infinitely many pairs of knots K_1 and K_2 such that*

$$u(K_1 \# K_2) < u(K_1) + u(K_2).$$

The proof of Theorem 1 begins with a concrete example. Let K_1 be the $(2,7)$–torus knot, and let K_2 be its mirror image. Then $u(K_1) = u(K_2) = 3$, while

$$u(K_1 \# K_2) \leq 5.$$

From this initial example, the authors produce infinitely many further counterexamples by combining their construction with earlier work on unknotting sequences.

References

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