The proof of two or infinity conjecture

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In a paper (Cristofaro-Gardiner et al. 2023) recently accepted for publication in the *Journal of the American Mathematical Society*, Cristofaro-Gardiner, Hryniewicz, Hutchings, and Liu proved the Hofer–Wysocki–Zehnder two-or-infinity conjecture, resolving a major open problem in Finsler geometry.

Finsler geometry generalizes Riemannian geometry while preserving many of its variational and dynamical aspects. Unlike the Riemannian setting, however, a Finsler metric may be anisotropic: the "cost" of moving in one direction need not agree with the cost of moving in the opposite direction. This additional flexibility makes Finsler geometry well suited to phenomena in which direction matters, including problems in optimal control, geometric optics, and dynamical systems.

To keep the discussion as concrete as possible, let us restrict attention to the two-sphere

$$S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}.$$

For each $x \in S^2$, the tangent space at x is

$$T_xS^2 = \{v \in \mathbb{R}^3 : x \cdot v = 0\},$$

and the tangent bundle is

$$TS^2 = \{(x, v) : x \in S^2,\ v \in T_xS^2\}.$$

A *Finsler metric* on S^2 is a function

$$F : TS^2 \to [0, \infty)$$

such that:

- F is continuous on all of TS^2 and smooth on $TS^2 \setminus \{(x, 0) : x \in S^2\}$;
- $F(x, v) = 0$ if and only if $v = 0$;
- $F(x, \lambda v) = \lambda F(x, v)$ for every $\lambda > 0$;
- for every $x \in S^2$ and every nonzero $v \in T_xS^2$, the bilinear form
$$g_{x,v}(u, w) = \frac{1}{2} \frac{\partial^2}{\partial s\, \partial t} F^2(x, v + su + tw)\Big|_{s=t=0}$$
is positive definite on T_xS^2.

Thus, a Finsler metric assigns a length to each tangent vector, much as a Riemannian metric does, but without requiring that this length arise from an inner product.

If $\gamma : [a,b] \to S^2$ is a smooth curve, its F-length and F-energy are defined by

$$L_F(\gamma) = \int_a^b F(\gamma(t), \dot\gamma(t))\,dt, \qquad E_F(\gamma) = \frac{1}{2}\int_a^b F(\gamma(t), \dot\gamma(t))^2\,dt.$$

A smooth curve γ with $\dot\gamma(t) \neq 0$ is called an *F-geodesic* if it is a critical point of the energy functional E_F under smooth variations with fixed endpoints.

A *closed geodesic* is a geodesic $\gamma : \mathbb{R} \to S^2$ for which there exists $T > 0$ such that

$$\gamma(t+T) = \gamma(t), \qquad \dot\gamma(t+T) = \dot\gamma(t) \qquad \text{for all } t \in \mathbb{R}.$$

Two closed geodesics are regarded as equivalent if they differ only by a shift of the time parameter: γ_1 and γ_2 are equivalent if there exists $t_0 \in \mathbb{R}$ such that

$$\gamma_2(t) = \gamma_1(t + t_0).$$

If η is a closed geodesic and $m \geq 2$, its *m-fold iterate* is the closed geodesic $\eta^{(m)}$ obtained by traversing the same loop m times. A closed geodesic is called *prime* if it is not an iterate of a shorter closed geodesic.

For closed surfaces equipped with a Riemannian metric, it has long been known that there are infinitely many prime closed geodesics. In the case of positive genus, this was established by Hadamard (Hadamard 1898). The remaining genus-zero case, namely the two-sphere, was later settled by Franks (Franks 1992). These results show that in Riemannian geometry closed geodesics are plentiful: no closed surface can have only finitely many prime closed geodesics.

For Finsler metrics, the picture is subtler. On surfaces of positive genus, the arguments from the Riemannian case can be adapted to prove the existence of infinitely many prime closed geodesics. The sphere, however, behaves in a genuinely different way. In a remarkable 1973 construction, Katok (Katok 1973) exhibited Finsler metrics on S^2 with exactly two prime closed geodesics. This example showed that the Riemannian phenomenon does not extend unchanged to the Finsler setting and led to the conjecture that no intermediate behavior is possible: a Finsler two-sphere should carry either exactly two or infinitely many prime closed geodesics.

In 2023, Cristofaro-Gardiner, Hryniewicz, Hutchings, and Liu posted the preprint (Cristofaro-Gardiner et al. 2023), confirming this long-standing conjecture.

Theorem 1 *Every Finsler metric on S^2 has either two or infinitely many prime closed geodesics.*

This theorem gives a strikingly sharp description of the global dynamics of geodesic flow on a Finsler two-sphere. Despite the flexibility of Finsler metrics, the theorem shows that the closed-geodesic structure is subject to a rigid dichotomy: one is in the exceptional Katok-type situation with exactly two prime closed geodesics, or else prime closed geodesics are forced to occur without bound.

More broadly, the result highlights the richness of the interaction between geometry, variational methods, and dynamical systems. It resolves a question that had shaped the field for decades and clarifies the special role played by the two-sphere in Finsler geometry. In this sense, the two-or-infinity theorem is not only a definitive answer to a classical problem, but also a landmark illustrating how subtle dynamical behavior can nevertheless obey a remarkably simple global principle.

References

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