The expected number of primes in shorter intervals

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The March 2026 issue of the *Annals of Mathematics* contains a paper by Guth and Maynard (Guth and Maynard 2026) proving that intervals of the form $[x, x+y]$ contain the expected number of primes for y as small as $x^{17/30+\epsilon}$. The result is both major and easy to state: it brings us significantly closer to the scale at which the distribution of primes is believed to behave regularly.

Euclid famously proved that there are infinitely many primes. The prime number theorem says, roughly, that near a large number x, the "density" of primes is about $1/\log x$. Thus, for any $\theta > 0$, an interval of length $l(x) = x^\theta$ located near x is expected to contain about

$$\frac{l(x)}{\log x}$$

primes. For instance, the distance between two consecutive squares $x = n^2$ and $(n+1)^2$ is

$$(n+1)^2 - n^2 = 2n + 1 \approx 2\sqrt{x},$$

so one expects roughly

$$\frac{2\sqrt{x}}{\log x}$$

primes between them.

There is, however, a striking gap between this heuristic prediction and what we can currently prove. In particular, we still cannot show that there is always even a single prime between two consecutive perfect squares. This is the famous Legendre conjecture.

A first major breakthrough came in 1930, when Hoheisel (Hoheisel 1930) proved the existence of large constants n_0 and d such that for every $n \geq n_0$ there is at least one prime between n^d and $(n+1)^d$. In fact, Hoheisel showed that one may take $d = 33000$. Since then, many authors have reduced the exponent d. For example, Johnston and Cully-Hugill (Johnston and Cully-Hugill 2024) proved that for every $n \geq 1$ there is a prime between n^{90} and $(n+1)^{90}$. The strength of their result is that it holds with $n_0 = 1$.

If one allows a large but explicit threshold n_0, then the exponent can be made much smaller. For example, Dudek (Dudek 2016) showed in 2016 that there is a prime between n^3 and $(n+1)^3$ for all

$$n \geq \exp(\exp(33.217)).$$

If one does not require an explicit threshold at all, then the exponent can be improved further. In 2001, Baker, Harman, and Pintz (Baker et al. 2001) proved that there exists a constant x_0 such that for all $x > x_0$, the interval $[x - x^{0.525}, x]$ contains at least one prime.

In fact, the proof in (Baker et al. 2001) yields more than mere existence: it implies that the interval $[x - x^{0.525}, x]$ contains at least

$$\frac{9}{100} \frac{x^{0.525}}{\log x}$$

primes for all $x > x_0$. This is of the same general shape as the expected main term

$$\frac{x^{0.525}}{\log x},$$

but still differs by the constant factor 0.09.

A much stronger type of result asks not merely for the existence of primes in short intervals, but for the *asymptotically correct number* of primes. In 1972, Huxley (Huxley 1972) proved that whenever $\theta > \frac{7}{12} = 0.5833\ldots$, an interval of length $l(x) = x^\theta$ contains

$$\approx \frac{l(x)}{\log x}$$

primes. This remained the state of the art for more than fifty years. Guth and Maynard (Guth and Maynard 2026) have now improved the exponent to

$$\theta > \frac{17}{30} = 0.5666\ldots.$$

Theorem 1 *For every $\epsilon > 0$ and every $y \in [x^{\frac{17}{30}+\epsilon}, x^{0.99}]$, the interval $[x, x+y]$ contains*

$$\frac{y}{\log x} + O_\epsilon\Big(y \exp\Big(-\sqrt[4]{\log x}\Big)\Big)$$

primes.

This theorem is exciting not only because it improves a longstanding record, but also because it narrows the gap between what analytic number theorists expect and what they can prove. The natural conjectural scale is far smaller, but pushing the asymptotic formula below Huxley's $\frac{7}{12}$ barrier after more than half a century is a remarkable advance. It is the kind of result that is easy to explain, hard to obtain, and likely to shape future work on primes in short intervals.

References

Baker, Roger C., Glyn Harman, and János Pintz. 2001. "The Difference Between Consecutive Primes, II." *Proc. Lond. Math. Soc.* 83 (3): 532–62.

Dudek, Adrian W. 2016. "An Explicit Result for Primes Between Cubes." *Funct. Approx. Comment. Math.* 55 (2): 177–97. https://doi.org/10.7169/facm/2016.55.2.3.

Guth, Larry, and James Maynard. 2026. "New Large Value Estimates for Dirichlet Polynomials." *Ann. Of Math. (2)* 203 (2). https://doi.org/10.4007/annals.2026.203.2.6.

Hoheisel, Guido. 1930. "Primzahlprobleme in Der Analysis." *Sitz. Preuss. Akad. Wiss.* 33: 3–11.

Huxley, M. N. 1972. "On the Difference Between Consecutive Primes." *Invent. Math.* 15: 164–70. https://doi.org/10.1007/BF01418933.

Johnston, Daniel R, and Michaela Cully-Hugill. 2024. "On the Error Term in the Explicit Formula of Riemann-von Mangoldt II." *arXiv Preprint arXiv:2402.04272*.