Harald Helfgott's talk on growth, expansion and escape in groups

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This post continues our series of online seminars devoted to accessible presentations of some of the most significant mathematical theorems of the 21st century. I am delighted to report that on March 4, Prof. Assaf Naor delivered a clear and engaging lecture on the sparsest cut problem. If you missed the talk, the recording is now available on YouTube.

The next lecture in the series will take place on

Wednesday, April 15, 2026, at 4:00 PM (UK time).

Prof. Harald Helfgott (Paris City University, France) will speak on the topic of *growth, expansion, and escape in groups*.

To join the talk, simply click here at the scheduled time—no registration is required. For a full list of upcoming seminars, please visit: https://th21.le.ac.uk/next-talks/.

For announcements of other talks, as well as descriptions of recent major mathematical breakthroughs, see the full list of posts on this blog.

We now recall some basic notions that will be relevant for the upcoming lecture.

A subset S of a group G is called *normal* if $gag^{-1} \in S$ for all $a \in S$ and $g \in G$. A group G is called *simple* if its only normal subgroups are $\{e\}$ and G itself.

The simplest examples of simple groups are the cyclic groups $\mathbb{Z}_p$ for prime p, consisting of the integers $\{0, 1, \dots, p-1\}$ with addition modulo p.

Another important class of examples is given by the alternating groups. Let $\mathrm{Sym}(n)$ denote the group of all bijections $f : [n] \to [n]$, where $[n] = \{1, 2, \dots, n\}$. This group, with composition as the operation, is called the *symmetric group*. For $f \in \mathrm{Sym}(n)$, let $M(f)$ be the number of pairs $1 \le i < j \le n$ such that $f(i) > f(j)$. The subgroup

$$\mathrm{Alt}(n) = \{f \in \mathrm{Sym}(n) : M(f) \text{ is even}\}$$

is called the *alternating group*, and it is simple for all $n \geq 5$.

We next consider groups of matrices. Let $\mathbb{F}_p$ be the finite field with p elements. The group $\mathrm{SL}_n(\mathbb{F}_p)$ consists of all $n \times n$ matrices with entries in $\mathbb{F}_p$ and determinant 1. Identifying matrices A and $-A$, one obtains the quotient group $\mathrm{PSL}_n(\mathbb{F}_p)$. For $n = 2$, the group $\mathrm{PSL}_2(\mathbb{F}_p)$ is simple for every prime $p \geq 5$.

A major achievement of 20th-century mathematics is the classification theorem for finite simple groups. It asserts that every finite simple group belongs to one of several explicitly described families (together with a finite list of exceptional cases). In particular, any such group is either cyclic of prime order, an alternating group, a group of Lie type such as $\mathrm{PSL}_n(\mathbb{F}_p)$, or one of the sporadic groups.

This classification is extremely powerful: to prove that all finite simple groups satisfy a given property, it suffices to verify it separately for each family. However, this reduction does not make all problems easy—many properties remain highly nontrivial even within a single family.

One important example concerns the *diameter* of a finite group. A subset $S \subset G$ is called a *generating set* if every element of G can be written as a finite product of elements of S and their inverses. Define $d(G, S)$ to be the smallest integer m such that every $g \in G$ can be expressed as a product of at most m elements of $S \cup S^{-1}$. The *diameter* of G is

$$\mathrm{diam}(G) = \max_S d(G, S),$$

where the maximum is taken over all generating sets S.

In 1992, Babai (Babai and Seress 1992) conjectured that there exist constants $M, c > 0$ such that

$$\mathrm{diam}(G) \leq M(\log |G|)^c \tag{1}$$

for every non-abelian finite simple group G.

Using the classification theorem, one may attempt to prove this conjecture case by case. A major breakthrough in this direction was achieved by Helfgott (Helfgott 2008) in 2008 for the groups $\mathrm{SL}_2(\mathbb{F}_p)$ and $\mathrm{PSL}_2(\mathbb{F}_p)$.

Theorem 1 (Helfgott, 2008) *There exist absolute constants $M, c > 0$ such that for every prime p, the bound (1) holds for $G = \mathrm{SL}_2(\mathbb{F}_p)$ and $G = \mathrm{PSL}_2(\mathbb{F}_p)$.*

The proof relies on a striking *growth phenomenon*: subsets of these groups expand rapidly under multiplication. A key ingredient is the following product theorem. If $A \subset G$ is not contained in a proper subgroup and satisfies $|A| < p^{3-\varepsilon}$, then

$$|A \cdot A \cdot A| \geq c|A|^{1+\delta}, \tag{2}$$

for constants $c, \delta > 0$ depending only on ε. On the other hand, sufficiently large sets generate the entire group in boundedly many steps. Combining these facts yields the diameter bound.

This work opened the door to further developments. In 2016, Pyber and Szabó (Pyber and Szabó 2016) established far-reaching generalizations, proving Babai's conjecture for all simple subgroups of $\mathrm{SL}_n(\mathbb{F}_p)$ for fixed n. A major remaining challenge is to obtain bounds that are *uniform* in n.

The case of alternating groups requires entirely different ideas, since product theorems such as (2) fail in this setting. For $G = \mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$, Babai and Seress (Babai and Seress 1988) showed in 1988 that

$$\mathrm{diam}(G) \leq \exp\big((1+o(1))\sqrt{\log|G|}\big),$$

a bound that remained unimproved for over 25 years.

In 2014, Helfgott and Seress (Helfgott and Seress 2014) achieved a significant breakthrough.

Theorem 2 (Helfgott–Seress, 2014) *There exists a constant $C > 0$ such that*

$$\mathrm{diam}(G) \leq \exp\big(C(\log n)^4 \log\log n\big)$$

for all n, where G is either $\mathrm{Alt}(n)$ *or* $\mathrm{Sym}(n)$.

In terms of $|G|$, this bound is of the form $\exp\big((\log\log|G|)^{O(1)}\big)$, which is called *quasi-polynomial* in $\log|G|$. Babai's conjecture predicts a polynomial bound in $\log|G|$, and this remains a major open problem.

Prof. Helfgott's upcoming lecture will provide an accessible introduction to these ideas and results, highlighting the role of growth and expansion in modern group theory.

References

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