On upper bounds for sphere packings

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This post continues my series on favorite theorems of the 21st century. For an overview of the categories and earlier selections, see this post.

My choice for 2001 in Geometry and Topology is the fundamental paper of Cohn and Elkies on sphere packing (Cohn and Elkies 2003). It introduced a new, general, and powerful method for proving upper bounds on sphere-packing densities, established improved upper bounds in every dimension $4 \leq d \leq 36$, and laid the groundwork for the eventual complete solution of the sphere-packing problem in dimensions $d = 8$ and $d = 24$.

We first recall the definition. Let

$$B(A, R) = \{X \in \mathbb{R}^d : |AX| < R\}$$

be an *open ball* in $\mathbb{R}^d$ with centre $A \in \mathbb{R}^d$ and radius $R > 0$. A *packing of congruent balls*, or *sphere packing*, is an (infinite) collection E of pairwise non-overlapping open balls of equal radius. The *upper density* of a packing E is defined by

$$\delta(E) = \limsup_{R \to \infty} \frac{|B(O, R) \cap E|}{|B(O, R)|}, \tag{1}$$

where $|\cdot|$ denotes d-dimensional volume. This quantity depends only on E and not on the choice of O. If the limit in (1) exists, then $\delta(E)$ is simply called the *density* of E. The sphere-packing problem in $\mathbb{R}^d$ asks for the largest possible upper density of a sphere packing.

The problem is trivial for $d = 1$, and the solution for $d = 2$ has long been known. In this previous post we discussed its resolution for $d = 3$, while in this previous post we discussed progress on its asymptotic behaviour as $d \to \infty$.

What happens when $d > 3$ is fixed? To prove a lower bound for the optimal density, it suffices to construct a good packing. Upper bounds are much subtler, because they must apply to all possible packings. In a paper submitted in 2001, Cohn and Elkies (Cohn and Elkies 2003) revolutionized the theory of upper bounds.

To formulate their result, we need one more definition. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *admissible* if there exist positive constants ϵ, C_1, C_2 such that

$$|f(x)| \leq \frac{C_1}{(1+\|x\|)^{d+\epsilon}} \quad \text{and} \quad |\hat{f}(x)| \leq \frac{C_2}{(1+\|x\|)^{d+\epsilon}} \quad \forall x \in \mathbb{R}^d,$$

where $\|x\|$ is the norm in $\mathbb{R}^d$, and $\hat{f}(x)$ is the Fourier transform defined as

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-2\pi i x t}\mathrm{d}t, \quad x \in \mathbb{R} \tag{2}$$

Now we are ready to state the Theorem:

Theorem 1 Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is an admissible function, is not identically zero, and satisfies the following conditions:

1. $f(x) \leq 0$ for $\|x\| \geq 1$;
2. $\hat{f}(t) \geq 0$ for all $t \in \mathbb{R}^d$.

Then the upper density of any sphere packing in $\mathbb{R}^d$ is bounded above by

$$\frac{f(0)}{2^d \hat{f}(0)} V_d,$$

where V_d is the volume of the unit ball in $\mathbb{R}^d$.

Let us call a function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying the hypotheses of Theorem 1 an *auxiliary function*. Theorem 1 gives a general method for proving upper bounds on sphere-packing densities: in each dimension d, one seeks an auxiliary function for which the ratio $\frac{f(0)}{2^d \hat{f}(0)}$ is as small as possible. In the same paper (Cohn and Elkies 2003), Cohn and Elkies used this method to derive improved upper bounds in dimensions $4 \leq d \leq 36$. In particular, for $d = 8$ they obtained the upper bound $1.000001\Delta_8$, where $\Delta_8 = \frac{\pi^4}{384}$ is the density of the well-known E_8 packing. Moreover, they conjectured the existence of a function $f^* : \mathbb{R}^8 \to \mathbb{R}$ which, when substituted into Theorem 1, yields the exact upper bound Δ_8. In 2017, Viazovska (Viazovska 2017) constructed such a function and thereby solved the sphere-packing problem in dimension $d = 8$. Shortly afterwards, Viazovska joined Cohn, Kumar, Miller, and Radchenko, and together they showed how the same method could also be used to solve the sphere-packing problem in dimension $d = 24$.

In dimensions d other than $1, 2, 3, 8$, and 24, the problem of determining the optimal sphere-packing upper density ρ_d^* remains open. It is conjectured that Theorem 1, even with an optimal auxiliary function, does not yield the sharp upper bound in any of the remaining open dimensions, so genuinely new ideas may be needed. Nevertheless, the Cohn–Elkies method remains the central

framework for the subject. In 2022, Cohn, de Laat, and Salmon (Cohn et al. 2022) developed a deeper version of this method that improved the previous best upper bounds on the optimal density in dimensions $4 \leq d \leq 7$ and $9 \leq d \leq 16$.

References

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