On uniform spacing of zeros of orthogonal polynomials

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The March 2026 issue of the Annals of Mathematics contains a paper by Eichinger, Lukić, and Simanek (Eichinger et al. 2026) establishing a near-optimal sufficient condition that guarantees the uniform spacing of zeros of orthogonal polynomials. This result resolves a long-standing open problem in the field.

Let μ be a probability measure on $\mathbb{R}$ with all moments finite,

$$\int_{-\infty}^{\infty} |\xi|^m \, \mathrm{d}\mu(\xi) < \infty, \qquad m = 0, 1, 2, \ldots.$$

For any polynomials p and q with real coefficients, the inner product

$$\langle p, q \rangle = \int_{-\infty}^{\infty} p(\xi) q(\xi) \, \mathrm{d}\mu(\xi)$$

is therefore well defined and finite.

A sequence of orthogonal polynomials is a family $p_0(\xi), p_1(\xi), p_2(\xi), \ldots$ such that $\deg p_n = n$ for all n and $\langle p_n, p_k \rangle = 0$ whenever $n \neq k$. For a given measure μ, such a sequence is unique up to normalization, that is, multiplication of each p_n by a nonzero constant c_n. For definiteness, we fix the normalization by requiring that each polynomial p_n has positive leading coefficient and satisfies $\langle p_n, p_n \rangle = 1$.

With this normalization, $p_0(\xi) = 1$, and for $n \geq 1$ the polynomials can be constructed via the Gram–Schmidt process. First we form the unnormalized orthogonal polynomial

$$\tilde{p}_n(\xi) = \xi^n - \sum_{i=0}^{n-1} \langle \xi^n, p_i \rangle \, p_i(\xi),$$

and then normalize it:

$$p_n(\xi) = \frac{\tilde{p}_n(\xi)}{\sqrt{\langle \tilde{p}_n, \tilde{p}_n \rangle}}.$$

It is well known that each polynomial p_n has exactly n real and simple zeros. Understanding the fine-scale distribution of these zeros along the real line is a central topic in the modern theory of orthogonal polynomials.

Fix a point $\xi \in \mathbb{R}$ and denote the zeros of p_n by $\xi_j^{(n)}$, indexed so that

$$\cdots < \xi_{-1}^{(n)} < \xi_0^{(n)} \le \xi < \xi_1^{(n)} < \xi_2^{(n)} < \cdots .$$

For each n, the values $\xi_j^{(n)}$ are defined for exactly n integers j. We say that the measure μ exhibits *clock behaviour* at ξ if, for every $j \in \mathbb{Z}$, the zero $\xi_j^{(n)}$ exists for all sufficiently large n, and there is a sequence $\tau_n \to \infty$ such that

$$\lim_{n\to\infty} \tau_n\big(\xi_{j+1}^{(n)} - \xi_j^{(n)}\big) = 1, \qquad j \in \mathbb{Z}. \tag{1}$$

Informally, this means that the zeros of p_n become asymptotically equally spaced near ξ.

A useful tool for studying this phenomenon is the Christoffel–Darboux (CD) kernel,

$$K_n(z,w) = \sum_{j=0}^{n-1} p_j(z)\overline{p_j(w)}, \qquad z,w \in \mathbb{C}, \quad n = 1,2,\ldots .$$

It was shown in (Levin and Lubinsky 2008) that (1) holds at $\xi \in \mathbb{R}$ provided that the CD kernel satisfies the scaling limit

$$\lim_{n\to\infty} \frac{1}{K_n(\xi,\xi)} K_n\left(\xi + \frac{z}{\tau_n},\, \xi + \frac{w}{\tau_n}\right) = \frac{\sin\big(\pi(\overline{w} - z)\big)}{\pi(\overline{w} - z)}, \qquad z,w \in \mathbb{C}. \tag{2}$$

The right-hand side is interpreted as 1 when $\overline{w} = z$.

The limiting kernel in (2) is independent of the measure μ. For this reason the phenomenon is known as *bulk universality*. It closely parallels the situation in random matrix theory, where the local statistics of eigenvalues converge to universal limits that depend only weakly on the distribution of matrix entries.

Over the past two decades many authors established (2) under various assumptions on μ. In 2026, Eichinger, Lukić, and Simanek (Eichinger et al. 2026) proved the following theorem, which unifies and significantly extends many earlier results.

Theorem 1 *Let μ be a probability measure on $\mathbb{R}$ with infinite (though not necessarily unbounded) support and finite moments. Assume that*

$$\lim_{n\to\infty} K_n(z,z) = \infty$$

for some (and therefore every) $z \in \mathbb{C}\setminus\mathbb{R}$. Let $\xi \in \mathbb{R}$ and suppose there exists $0 < \eta < \infty$ such that

$$\eta = \lim_{\epsilon\to 0^+} \frac{\mu((\xi-\epsilon,\xi))}{\epsilon} = \lim_{\epsilon\to 0^+} \frac{\mu([\xi,\xi+\epsilon))}{\epsilon}. \tag{3}$$

Then (2) holds with $\tau_n = \eta K_n(\xi,\xi)$, uniformly on compact subsets of $(z,w) \in \mathbb{C}\times\mathbb{C}$.

Thus Theorem 1 establishes bulk universality (2), and hence clock behaviour (1), for a broad class of measures μ. Moreover, subsequent work (Eichinger et al. 2024) shows that the theorem is essentially optimal: the condition (3) is not only sufficient but also almost necessary for bulk universality. In particular, (3) implies

$$\lim_{n\to\infty} \frac{K_{n+1}(\xi,\xi)}{K_n(\xi,\xi)} = 1, \tag{4}$$

while conversely the conclusion of Theorem 1, together with (4), implies (3).

References

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