The hot spot conjecture for lip domains

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This post continues my series on favorite theorems of the 21st century. For an overview of the categories and earlier selections, see this post.

My choice for 2001 in Analysis is the proof of *hot spots conjecture* for lip domains by Atar and Burdzy, a result I have already mentioned in this previous blog post.

Recall that the celebrated *hot spots conjecture* predicts that for a flat, insulated plate of uniform material, the points that remain the hottest and coldest for the longest duration must eventually migrate to the boundary of the domain. It serves as a bridge between the spectral theory of the Laplacian and the physical intuition of heat diffusion. Despite its intuitive appeal, the conjecture has proven to be mathematically elusive, revealing deep connections between domain geometry and the nodal sets of eigenfunctions.

To state the conjecture rigorously, we need to consider the Laplacian operator $\Delta = \sum \partial^2_{x_i}$ acting on a suitable class of domains. Let $\Omega \subset \mathbb{R}^2$ be a *Lipschitz domain*—a bounded domain where the boundary $\partial\Omega$ is locally representable as the graph of a Lipschitz continuous function. Such a regularity assumption ensures that the outward unit normal vector n is defined almost everywhere.

The physical behavior of heat $v(x,t)$ in an insulated domain is governed by the heat equation $\partial_t v = \Delta v$ with homogeneous Neumann boundary conditions $\frac{\partial v}{\partial n} = 0$. As $t \to \infty$, the solution $v(x,t)$ is dominated by the first non-trivial term in its spectral expansion, which corresponds to the *second Neumann eigenvalue* $\mu_2 = \mu_2(\Omega)$. This eigenvalue is defined as the smallest positive real number satisfying the Helmholtz equation:

$$\begin{cases} -\Delta u = \mu_2 u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The solutions $u : \Omega \to \mathbb{R}$ to this system are known as *second Neumann eigenfunctions*. Because the first eigenfunction $\mu_1 = 0$ is constant, the second eigenfunction dictates the primary spatial distribution of heat as the system approaches equilibrium.

Proposed by Jeffrey Rauch in 1974–1975, the conjecture asserts that any second Neumann eigenfunction u attains its maximum and minimum exclusively on the boundary $\partial\Omega$. Specifically:

$$\{x \in \overline{\Omega} : |u(x)| = \sup_{y\in\Omega} |u(y)|\} \subset \partial\Omega. \tag{2}$$

For decades, the conjecture was suspected to hold for all bounded domains. However, in 1999, Burdzy and Werner (Burdzy and Werner 1999) provided a surprising counterexample using a planar domain with multiple "inner" boundaries (holes). This demonstrated that the presence of non-trivial topology or extreme non-convexity could force the "hot spots" into the interior. Consequently, modern research has shifted toward identifying the specific geometric constraints—such as simple connectivity or convexity—under which the conjecture remains valid.

A landmark result in this direction was established by Atar and Burdzy in a paper (Atar and Burdzy 2004) submitted in 2001. They utilized probabilistic methods, specifically the coupling of reflected Brownian motions, to prove the conjecture for a specific class of "lip-domains."

Theorem 1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, connected Lipschitz domain, given by*

$$\Omega = \{(x_1, x_2) : f_1(x_1) < x_2 < f_2(x_1)\},$$

where f_1, f_2 are Lipschitz functions with constant 1. Then every Neumann eigenfunction corresponding to $\mu_2(\Omega)$ attains its maximum and minimum at boundary points only.

As mentioned in mentioned in this previous blog post, the hot spot conjecture was later proved for some other domains, most notably for all triangles. However, triangles are rather special shapes, while Theorem 1 covers a lot of various shapes, and, in my opinion, Theorem 1 provides the current most significant step toward the broader proof for all convex planar domains, which remains one of the most intriguing open problems in spectral geometry.

References

Atar, Rami, and Krzysztof Burdzy. 2004. “On Neumann Eigenfunctions in Lip Domains.” *J. Amer. Math. Soc.* 17 (2): 243–65.

Burdzy, Krzysztof, and Wendelin Werner. 1999. “A Counterexample to the ‘Hot Spots’ Conjecture.” *Ann. Of Math.* 149 (1): 309–17.