On the spectrum of quasi-periodic Schrödinger operator

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The March 2026 issue of the Annals of Mathematics contains a paper by Karpeshina, Parnovski and Shterenberg (Karpeshina, Parnovski, and Shterenberg 2026) proving that the spectrum of a generic quasi-periodic Schrödinger operator contains a semi-axis. Roughly speaking, their result shows that in many quasi-periodic quantum systems there are no spectral gaps at sufficiently high energies.

To describe the setting, fix an integer $d \geq 2$. Let

$$L^2(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{C} \text{ measurable } : \int_{\mathbb{R}^d} |f(x)|^2 \, dx < \infty \right\},$$

the space of square-integrable functions on $\mathbb{R}^d$, where functions that agree almost everywhere are identified. We also consider the Sobolev space $H^2(\mathbb{R}^d)$, consisting of functions in $L^2(\mathbb{R}^d)$ whose first and second partial derivatives (in the weak sense) also belong to $L^2(\mathbb{R}^d)$. Informally, $H^2(\mathbb{R}^d)$ is the natural class of functions on which the Laplacian can act.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a bounded real-valued function, called the *potential*. The associated Schrödinger operator is

$$H := -\Delta + V,$$

which acts as an operator

$$H : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

defined by

$$(Hf)(x) = -\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}(x) + V(x) f(x), \qquad \text{for almost every } x = (x_1, \ldots, x_d) \in \tag{1}$$

This operator is the standard mathematical model for a quantum particle moving in $\mathbb{R}^d$ under the influence of the potential V.

A central object of study is the *spectrum* of H, which describes the possible energy levels of the system. Formally, the resolvent set $\rho(H)$ consists of those $\lambda \in \mathbb{C}$ for which the operator $H - \lambda I$ is invertible and its inverse

$$(H - \lambda I)^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is bounded. The spectrum is the complement

$$\sigma(H) := \mathbb{C} \setminus \rho(H).$$

Because V is bounded and real-valued, the operator H is self-adjoint and its spectrum lies on the real line.

One of the classical questions in spectral theory is whether the spectrum eventually fills out a whole half-line. More precisely, does there exist $\lambda_* \in \mathbb{R}$ such that

$$[\lambda_*, \infty) \subseteq \sigma(H)?$$

When this happens, we say that H has the *Bethe–Sommerfeld property*. In physical terms, this means that above some energy threshold there are no spectral gaps: every sufficiently large energy is allowed.

Karpeshina, Parnovski and Shterenberg proved this property for a broad family of quasi-periodic potentials. To construct such potentials, fix an integer $l > d$ and choose vectors

$$\omega_1, \dots, \omega_l \in \mathbb{R}^d,$$

called the *basic frequencies*. It is convenient to collect them into a single vector

$$\vec{\omega} := (\omega_1, \dots, \omega_l) \in (\mathbb{R}^d)^l \cong \mathbb{R}^{dl}.$$

For $n = (n_1, \dots, n_l) \in \mathbb{Z}^l$ we write

$$|n| := \max_{1 \le j \le l} |n_j| \qquad \text{and} \qquad n\vec{\omega} := \sum_{j=1}^{l} n_j \omega_j \in \mathbb{R}^d.$$

For vectors $\theta, x \in \mathbb{R}^d$, denote by

$$\langle \theta, x \rangle := \sum_{j=1}^{d} \theta_j x_j$$

the standard Euclidean scalar product.

Fix $Q \in \mathbb{N}$. For each $n \in \mathbb{Z}^l$ with $|n| < Q$, choose a complex coefficient V_n, subject to the symmetry condition

$$V_{-n} = \overline{V_n}.$$

Define the function

$$V_{\vec{\omega}}(x) = \sum_{|n|<Q} V_n e^{i\langle n\vec{\omega}, x \rangle}, \qquad x \in \mathbb{R}^d. \tag{2}$$

Because the sum is finite, $V_{\vec{\omega}}$ is bounded. The relation $V_{-n} = \overline{V_n}$ ensures that the function is real-valued. Potentials of this form are called *(finite) quasi-periodic potentials*. They generalize periodic potentials by allowing several independent frequencies that do not necessarily fit into a single lattice.

The main result of (Karpeshina, Parnovski, and Shterenberg 2026) can be stated as follows.

Theorem 1 *Fix $Q \in \mathbb{N}$ and coefficients $\{V_n\}_{|n|<Q}$ satisfying $V_{-n} = \overline{V_n}$. Then there exists a set*

$$\Omega_* \subset \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^{dl}$$

of full Lebesgue measure 1 such that for every $\vec{\omega} \in \Omega_*$, if $V_{\vec{\omega}}$ is given by (2) and

$$H_{\vec{\omega}} = -\Delta + V_{\vec{\omega}}$$

is the corresponding Schrödinger operator (1), then there exists a sufficiently large number

$$\lambda_* = \lambda_*(\vec{\omega}, \{V_n\}) \in \mathbb{R}$$

for which

$$[\lambda_*, \infty) \subset \sigma(H_{\vec{\omega}}).$$

The phrase "full Lebesgue measure" means that the exceptional set of frequencies is negligible: almost every choice of frequencies works.

The theorem shows that for a generic choice of basic frequencies the spectrum cannot have infinitely many gaps accumulating at high energies. Beyond some threshold λ_*, every energy belongs to the spectrum. In other words, the high-energy part of the spectrum eventually becomes gapless.

This phenomenon is the quasi-periodic analogue of the classical Bethe–Sommerfeld theorem for periodic Schrödinger operators. The result shows that even when periodicity is replaced by the more complicated quasi-periodic structure, multidimensional systems still display a robust high-energy spectral regime. The full theorem in (Karpeshina, Parnovski, and Shterenberg 2026) is actually stronger: it proves that the same semi-axis lies not just in $\sigma(H_{\vec{\omega}})$, but in the *absolutely continuous spectrum*. This refinement concerns a finer decomposition of the spectrum and is more technical to define, but it implies particularly stable transport properties for the corresponding quantum system.

References

Karpeshina, Yulia, Leonid Parnovski, and Roman Shterenberg. 2026. "Bethe–Sommerfeld Conjecture and Absolutely Continuous Spectrum of Multi-Dimensional Quasi-Periodic Schrödinger Operators." *Ann. Of Math. (2)* 203 (2). https://doi.org/10.4007/annals.2026.203.2.1.