

On the spectrum of quasi-periodic Schrödinger operator

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The March 2026 issue of the Annals of Mathematics contains a paper by Karpeshina, Parnowski and Shterenberg (Karpeshina et al. 2026) proving that the spectrum of a generic quasi-periodic Schrödinger operator contains a semi-axis. Roughly speaking, their result shows that in many quasi-periodic quantum systems there are no spectral gaps at sufficiently high energies.

To describe the setting, fix an integer $d \geq 2$. Let

$$L^2(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{C} \text{ measurable} : \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty \right\},$$

the space of square-integrable functions on \mathbb{R}^d , where functions that agree almost everywhere are identified. We also consider the Sobolev space $H^2(\mathbb{R}^d)$, consisting of functions in $L^2(\mathbb{R}^d)$ whose first and second partial derivatives (in the weak sense) also belong to $L^2(\mathbb{R}^d)$. Informally, $H^2(\mathbb{R}^d)$ is the natural class of functions on which the Laplacian can act.

Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded real-valued function, called the *potential*. The associated Schrödinger operator is

$$H := -\Delta + V,$$

which acts as an operator

$$H : H^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

defined by

$$(Hf)(x) = - \sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}(x) + V(x)f(x), \quad \text{for almost every } x = (x_1, \dots, x_d) \in \mathbb{R}^d \quad (1)$$

This operator is the standard mathematical model for a quantum particle moving in \mathbb{R}^d under the influence of the potential V .

A central object of study is the *spectrum* of H , which describes the possible energy levels of the system. Formally, the resolvent set $\rho(H)$ consists of those $\lambda \in \mathbb{C}$ for which the operator $H - \lambda I$ is invertible and its inverse

$$(H - \lambda I)^{-1} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

is bounded. The spectrum is the complement

$$\sigma(H) := \mathbb{C} \setminus \rho(H).$$

Because V is bounded and real-valued, the operator H is self-adjoint and its spectrum lies on the real line.

One of the classical questions in spectral theory is whether the spectrum eventually fills out a whole half-line. More precisely, does there exist $\lambda_* \in \mathbb{R}$ such that

$$[\lambda_*, \infty) \subseteq \sigma(H)?$$

When this happens, we say that H has the *Bethe–Sommerfeld property*. In physical terms, this means that above some energy threshold there are no spectral gaps: every sufficiently large energy is allowed.

Karpeshina, Parnovski and Shterenberg proved this property for a broad family of quasi-periodic potentials. To construct such potentials, fix an integer $l > d$ and choose vectors

$$\omega_1, \dots, \omega_l \in \mathbb{R}^d,$$

called the *basic frequencies*. It is convenient to collect them into a single vector

$$\vec{\omega} := (\omega_1, \dots, \omega_l) \in (\mathbb{R}^d)^l \cong \mathbb{R}^{dl}.$$

For $n = (n_1, \dots, n_l) \in \mathbb{Z}^l$ we write

$$|n| := \max_{1 \leq j \leq l} |n_j| \quad \text{and} \quad n\vec{\omega} := \sum_{j=1}^l n_j \omega_j \in \mathbb{R}^d.$$

For vectors $\theta, x \in \mathbb{R}^d$, denote by

$$\langle \theta, x \rangle := \sum_{j=1}^d \theta_j x_j$$

the standard Euclidean scalar product.

Fix $Q \in \mathbb{N}$. For each $n \in \mathbb{Z}^l$ with $|n| < Q$, choose a complex coefficient V_n , subject to the symmetry condition

$$V_{-n} = \overline{V_n}.$$

Define the function

$$V_{\vec{\omega}}(x) = \sum_{|n| < Q} V_n e^{i\langle n\vec{\omega}, x \rangle}, \quad x \in \mathbb{R}^d. \quad (2)$$

Because the sum is finite, $V_{\vec{\omega}}$ is bounded. The relation $V_{-n} = \overline{V_n}$ ensures that the function is real-valued. Potentials of this form are called *(finite) quasi-periodic potentials*. They generalize periodic potentials by allowing several independent frequencies that do not necessarily fit into a single lattice.

The main result of (Karpeshina et al. 2026) can be stated as follows.

Theorem 1 Fix $Q \in \mathbb{N}$ and coefficients $\{V_n\}_{|n| < Q}$ satisfying $V_{-n} = \overline{V_n}$. Then there exists a set

$$\Omega_* \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^{dl}$$

of full Lebesgue measure 1 such that for every $\vec{\omega} \in \Omega_*$, if $V_{\vec{\omega}}$ is given by (2) and

$$H_{\vec{\omega}} = -\Delta + V_{\vec{\omega}}$$

is the corresponding Schrödinger operator (1), then there exists a sufficiently large number

$$\lambda_* = \lambda_*(\vec{\omega}, \{V_n\}) \in \mathbb{R}$$

for which

$$[\lambda_*, \infty) \subset \sigma(H_{\vec{\omega}}).$$

The phrase “full Lebesgue measure” means that the exceptional set of frequencies is negligible: almost every choice of frequencies works.

The theorem shows that for a generic choice of basic frequencies the spectrum cannot have infinitely many gaps accumulating at high energies. Beyond some threshold λ_* , every energy belongs to the spectrum. In other words, the high-energy part of the spectrum eventually becomes gapless.

This phenomenon is the quasi-periodic analogue of the classical Bethe–Sommerfeld theorem for periodic Schrödinger operators. The result shows that even when periodicity is replaced by the more complicated quasi-periodic structure, multidimensional systems still display a robust high-energy spectral regime. The full theorem in (Karpeshina et al. 2026) is actually stronger: it proves that the same semi-axis lies not just in $\sigma(H_{\vec{\omega}})$, but in the *absolutely continuous spectrum*. This refinement concerns a finer decomposition of the spectrum and is more technical to define, but it implies particularly stable transport properties for the corresponding quantum system.

References

- Karpeshina, Yulia, Leonid Parnovski, and Roman Shterenberg. 2026. “Bethe–Sommerfeld Conjecture and Absolutely Continuous Spectrum of Multi-Dimensional Quasi-Periodic Schrödinger Operators.” *Ann. Of Math. (2)* 203 (2). <https://doi.org/10.4007/annals.2026.203.2.1>.