The Gaussian correlation inequality and its generalization

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In February 2026, Milman, Nakamura and Tsuji posted online a preprint (Milman, Nakamura, and Tsuji 2026) in which they proved a natural and conceptually appealing strengthening of the famous *Gaussian correlation inequality*. The result fits into a long line of research on correlation phenomena for Gaussian measures, a topic that lies at the intersection of probability theory, convex geometry, and functional analysis.

By definition, events A and B are *independent* if

$$\mu(A \cap B) = \mu(A) \cdot \mu(B),$$

where μ is the underlying probability measure. In general, independence is a very strong requirement, and most events are not independent. It is therefore natural to ask whether, under suitable structural assumptions, one can guarantee that two events are at least *positively correlated*, meaning that

$$\mu(A \cap B) \geq \mu(A) \cdot \mu(B). \quad (1)$$

For arbitrary events this inequality need not hold, but remarkable additional structure emerges when the measure and the sets possess convexity and symmetry properties.

The *Gaussian correlation conjecture* predicted that inequality (1) holds for all convex and centrally symmetric sets $A, B \subset \mathbb{R}^n$ when μ is the n-dimensional Gaussian probability measure on $\mathbb{R}^n$, centred at the origin[1]. In other words, the conjecture asserted that convex symmetric sets under Gaussian measure are always positively correlated.

A special case of the conjecture was formulated by Dunnett and Sobel in 1955. The general form was later stated by Gupta, Eaton, Olkin, Perlman, Savage and Sobel in 1972 (Gupta et al. 1972). Despite its simple and intuitive formulation, the conjecture resisted proof for more than forty years and became one of the most prominent open problems in Gaussian probability. Numerous partial results were obtained, often under additional restrictions on the sets involved, and the problem attracted attention from researchers in several areas of mathematics.

The conjecture was finally resolved in 2014 by Royen (Royen 2014), whose proof was both surprisingly short and technically ingenious. The result is now known as the Gaussian correlation inequality.

Theorem 1 *Let μ be the n-dimensional Gaussian probability measure on $\mathbb{R}^n$, centred at the origin. Then inequality (1) holds for all convex and centrally symmetric sets $A, B \subset \mathbb{R}^n$.*

Royen's argument is notable not only for settling the conjecture but also for its generality. The proof actually works in a broader setting, namely for multivariate gamma distributions.

The recent preprint of Milman, Nakamura and Tsuji (Milman, Nakamura, and Tsuji 2026) goes a step further by strengthening inequality (1). They prove that under suitable conditions one has

$$\mu(A \cap B)\, \mu(A + B) \geq \mu(A) \cdot \mu(B), \tag{2}$$

where

$$A + B = \{x + y : \, x \in A, \; y \in B\}$$

denotes the Minkowski sum of the sets A and B.

This inequality refines the classical Gaussian correlation inequality in several ways. First, since $\mu(A + B) \leq 1$, inequality (2) immediately implies (1). Second, the new form is aesthetically appealing because the Gaussian measure μ appears exactly twice on each side of the inequality, giving it a balanced multiplicative structure. From a geometric viewpoint, the appearance of the Minkowski sum is also natural: it connects the problem to classical themes in convex geometry, where Minkowski addition plays a central role.

Perhaps even more strikingly, the authors show that inequality (2) remains valid even without central symmetry. More precisely, the result holds whenever A and B are convex sets satisfying the centering conditions

$$\int_A x \, d\mu(x) = 0, \qquad \int_B x \, d\mu(x) = 0.$$

Thus the symmetry assumption in the classical Gaussian correlation inequality can be replaced by a weaker requirement that each set has Gaussian barycenter at the origin.

This new result highlights once again the deep and somewhat mysterious interaction between Gaussian measures and convex geometry. Even after the resolution of the Gaussian correlation conjecture, the subject continues to reveal richer structures and stronger inequalities, suggesting that further developments are still to come.

References

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1. Recall that a symmetric $n \times n$ matrix Σ is called *positive definite* if $x^T \Sigma x > 0$ for all non-zero $x \in \mathbb{R}^n$. Given a positive definite $n \times n$ matrix Σ, let $f : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$f(x) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right), \qquad x \in \mathbb{R}^n.$$

For a subset $A \subset \mathbb{R}^n$, define $\mu(A) = \int_A f(x)\, dx$.↩