Diophantine approximation on manifolds

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In February 2026, Beresnevich, Datta and Yang posted online a preprint (Beresnevich, Datta, and Yang 2026) in which they completed a decades-long program aimed at understanding how well points lying on curved spaces (so-called manifolds) can be approximated by rational points.

In this previous blog post we discussed Diophantine approximation and its foundational result—Khintchine's Theorem. It states that if $\psi : \mathbb{N} \to \mathbb{R}^+$ is a function such that $\sum_{n=1}^{\infty} \psi(n) = \infty$ and the sequence $n\psi(n)$ is non-increasing, then for almost every $\alpha \in \mathbb{R}$ there exist infinitely many pairs of integers m, n such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, \tag{1}$$

and, conversely, if $\sum_{n=1}^{\infty} \psi(n) < \infty$, then for almost every $\alpha \in \mathbb{R}$ there are at most finitely many pairs of integers m, n satisfying (1). In the previous post we also explained that an analogous statement remains valid when α is restricted to certain subsets of $\mathbb{R}$ with fractal structure, such as Cantor's middle-third set.

In 1962, Gallagher (Gallagher 1962) extended Khintchine's Theorem to the setting of simultaneous approximation of several real numbers. Let $\psi_1, \dots, \psi_d : \mathbb{R}^+ \to (0, 1)$ be non-increasing functions. We say that a point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ is simultaneously $(\psi_1, \dots, \psi_d)$-approximable if the inequalities

$$\left|x_i - \frac{m_i}{n}\right| < \frac{\psi_i(n)}{n} \qquad \text{for } 1 \le i \le d$$

are satisfied for infinitely many integer tuples $(n, m_1, \dots, m_d) \in \mathbb{N} \times \mathbb{Z}^d$. In the special case where $\psi_1 = \cdots = \psi_d = \psi$, we say that x is *simultaneously ψ-approximable*. Gallagher's theorem asserts that almost all $x \in \mathbb{R}^d$ are simultaneously $(\psi_1, \dots, \psi_d)$-approximable if

$$\mathrm{Sum}(\psi_1, \dots, \psi_d) := \sum_{n=1}^{\infty} \psi_1(n) \cdots \psi_d(n) = \infty,$$

and, conversely, that almost all $x \in \mathbb{R}^d$ fail to be simultaneously $(\psi_1, \dots, \psi_d)$-approximable if $\mathrm{Sum}(\psi_1, \dots, \psi_d) < \infty$.

Now imagine that, for certain functions ψ_1, ψ_2, we are interested in whether the pair of real numbers $\alpha = x$ and $\beta = x^2$ is simultaneously (ψ_1, ψ_2)-approximable for almost all x. The set of such pairs forms a one-dimensional curve inside $\mathbb{R}^2$, and hence has Lebesgue measure zero in $\mathbb{R}^2$. As a result, Gallagher's theorem—which concerns almost every point in the ambient space $\mathbb{R}^d$—is no longer applicable.

The recent paper of Beresnevich, Datta and Yang answers this question for all nondegenerate submanifolds of $\mathbb{R}^d$. A map $f : U \to \mathbb{R}^d$, defined on an open subset $U \subset \mathbb{R}^k$, is called nondegenerate if for almost every point $x_0 \in U$ there exists a positive integer l such that f is l times continuously differentiable in a neighbourhood of x_0 and the partial derivatives of f at x_0 of orders up to l span $\mathbb{R}^d$. The immersed manifold $M := f(U)$ is called nondegenerate if the immersion f is nondegenerate in this sense. Informally, this condition excludes manifolds that are too "flat" or contained in proper affine subspaces.

In 2025, Beresnevich and Datta (Beresnevich and Datta 2025) considered the important and well-studied special case $\psi_1 = \cdots = \psi_d = \psi$, and proved the following complete characterization.

Theorem 1 *Let $d \geq 2$, let M be any nondegenerate submanifold of $\mathbb{R}^d$, and let $\psi : \mathbb{R}^+ \to (0,1)$ be a non-increasing function. Then almost all points on M are simultaniously ψ-approximable if the series*

$$\sum_{n=1}^{\infty} \psi^d(n) \tag{2}$$

diverges, and almost no points on M are simultaniously ψ-approximable if the series (2) converges.

In their February 2026 preprint (Beresnevich, Datta, and Yang 2026), Beresnevich, Datta and Yang resolved the general case, allowing the functions ψ_i to differ.

Theorem 2 *Let $d \geq 2$, let M be any nondegenerate submanifold of $\mathbb{R}^d$, and let $\psi_1, \ldots, \psi_d : \mathbb{R}^+ \to (0,1)$ be non-increasing functions. Then almost all points on M are simultaniously $(\psi_1, \ldots, \psi_d)$-approximable if $\mathrm{Sum}(\psi_1, \ldots, \psi_d) = \infty$, and almost no points on M are simultaniously $(\psi_1, \ldots, \psi_d)$-approximable if $\mathrm{Sum}(\psi_1, \ldots, \psi_d) < \infty$.*

Theorems 1 and 2 are monumental achievements that culminate a long sequence of works containing many partial results. They provide a definitive metric theory of simultaneous Diophantine approximation on nondegenerate manifolds, settling problems that have guided the field for several decades.

References

Beresnevich, Victor, and Shreyasi Datta. 2025. “Rational Points Near Manifolds and Khintchine Theorem.” *arXiv Preprint arXiv:2505.01227*.

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