On prime values and prime factors of polynomials

Bogdan Grechuk • 25 Feb 2026

In a recent breakthrough, Pascadi (Pascadi 2026) proved that the greatest prime factor of x^2+1 exceeds $x^{1.3}$ for infinitely many integers x. This is the strongest result to date toward the famous fourth problem of Landau, which predicts that the polynomial x^2+1 takes prime values infinitely often.

Many of the deepest achievements in modern analytic number theory can be viewed as instances of the following overarching problem.

Question 1 *Let $P_i(x_1,\dots,x_n)$, $i=1,\dots,k$, be polynomials in n variables with integer coefficients. Under what conditions on the P_i are there infinitely many integer tuples $(x_1,\dots,x_n)$ such that all the values*

$$P_1(x_1,\dots,x_n),\dots,P_k(x_1,\dots,x_n)$$

are simultaneously prime?

This formulation encompasses a remarkable range of classical and modern results.

The story begins in 1837 with Dirichlet's theorem on arithmetic progressions: if a and b are coprime integers, then the linear polynomial $ax+b$ takes prime values for infinitely many integers x. This completely resolves Question 1 in the case of a single linear polynomial in one variable. It already illustrates a key principle: a simple necessary condition (here, $\gcd(a,b)=1$) is also sufficient.

The next layer of complexity appears with quadratic polynomials. Fermat asserted—and Euler proved—that every prime $p \equiv 1 \pmod 4$ can be written as

$$p = x^2 + y^2$$

for suitable integers x, y. Combined with Dirichlet's theorem, this shows that the quadratic form x^2+y^2 represents infinitely many primes.

Euler also established that there are infinitely many primes of the form x^2+2y^2 and x^2+3y^2. Later, Dirichlet settled the case $k=1$ of Question 1 for binary quadratic forms

$$P(x, y) = ax^2 + bxy + cy^2,$$

and Iwaniec extended this to all irreducible quadratic polynomials in two variables that genuinely depend on both variables.

For polynomials of degree at least 3, progress proved much more difficult. A landmark result came in 1998, when Friedlander and Iwaniec (Friedlander and Iwaniec 1998) showed that the quartic form

$$x^2 + y^4$$

takes prime values infinitely often. This was the first example of a genuinely nonlinear, higher-degree polynomial in two variables known to produce infinitely many primes.

Shortly thereafter, Heath-Brown (Heath-Brown 2001) established the same for

$$x^3 + 2y^3,$$

and Heath-Brown and Moroz (Heath-Brown and Moroz 2002) extended the method to general irreducible cubic forms

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

with $\gcd(a, b, c, d) = 1$, provided that the form is not identically even. These works introduced powerful refinements of the Hardy–Littlewood circle method and sieve techniques.

In 2017, Heath-Brown and Li (Heath-Brown and Li 2017) proved that there are infinitely many primes of the form

$$x^2 + p^4,$$

where x is an integer and p is prime. This answers Question 1 for the pair of polynomials

$$P_1(x, y) = x^2 + y^4, \qquad P_2(x, y) = y,$$

requiring both P_1 and P_2 to be prime.

A dramatic advance occurred in 2025, when Green and Sawhney (Green and Sawhney 2024) proved the first result in which *both* variables are required to be prime.

Theorem 1 *If $n \equiv 0$ or $4 \pmod 6$, then there exist infinitely many primes of the form*

$$x^2 + ny^2$$

with both x and y prime.

They further observed that the same method should apply to general positive definite binary quadratic forms $ax^2 + bxy + cy^2$ with $b^2 - 4ac < 0$, provided that there is no fixed prime divisor of $xy(ax^2 + bxy + cy^2)$. This effectively settles Question 1 for three polynomials

$$P_1 = ax^2 + bxy + cy^2, \qquad P_2 = x, \qquad P_3 = y$$

under the stated hypotheses.

Question 1 also encompasses longer patterns of primes. In 2008, Green and Tao (Green and Tao 2008) proved that for every $k \geq 1$ there are infinitely many arithmetic progressions of length k consisting entirely of primes. This settles Question 1 for

$$P_i(x, y) = x + (i - 1)y, \qquad i = 1, \ldots, k,$$

with $y \neq 0$.

In the same year, Tao and Ziegler (Tao and Ziegler 2008) extended this to polynomial progressions

$$x + P_1(m), \ldots, x + P_k(m),$$

for arbitrary integer polynomials P_i satisfying $P_i(0) = 0$. These results required a profound synthesis of additive combinatorics, ergodic theory, and analytic number theory.

Despite this extraordinary progress, many elementary-looking instances of Question 1 remain unresolved.

- The case $P_1 = x$, $P_2 = x + 2$ is the twin prime conjecture.
- The case $P_1 = x$, $P_2 = 2x + 1$ is the conjecture on Sophie Germain primes.
- Even the single-polynomial case $k = 1$, $n = 1$ is open for every nonlinear polynomial $P(x)$.

A particularly famous example is Landau's fourth problem, which asks whether

$$x^2 + 1$$

is prime for infinitely many integers x.

Although we cannot yet prove that $x^2 + 1$ is prime infinitely often, we can study its factorization. In 2010, Friedlander and Iwaniec (Friedlander and Iwaniec 2010, Theorem 25.9) showed that $x^2 + 1$ is the product of two primes for infinitely many x. Obviously, the larger of two prime factors must be exceed x.

In fact, much earlier, Chebyshev (with the first published proof due to Markov in 1895) proved that for every constant $c > 0$, the greatest prime factor of $x^2 + 1$ exceeds cx for infinitely many x. Over more than a century, this linear bound was steadily improved.

Before 2026, the strongest result was due to Merikoski (Merikoski 2023), who proved that the greatest prime factor exceeds $x^{1.279}$ infinitely often. Pascadi (Pascadi 2026) has now pushed this further.

Theorem 2 *For infinitely many integers x, the greatest prime factor of x^2+1 exceeds $x^{1.3}$.*

While this still falls short of proving that $x^2 + 1$ is itself prime infinitely often, it represents the most substantial quantitative progress so far toward Landau's conjecture.

References

Friedlander, John, and Henryk Iwaniec. 1998. "The Polynomial Captures Its Primes." *Ann. of Math.* 148: 945–1040.

Friedlander, John, and Henryk Iwaniec. 2010. *Opera de Cribro*. Vol. 57. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI. https://doi.org/10.1090/coll/057.

Green, Ben, and Mehtaab Sawhney. 2024. "Primes of the Form." *arXiv Preprint arXiv:2410.04189*.

Green, Ben, and Terence Tao. 2008. "The Primes Contain Arbitrarily Long Arithmetic Progressions." *Ann. of Math.* 167 (2): 481–547.

Heath-Brown, D. R. 2001. "Primes Represented By." *Acta Math.* 186 (1): 1–84.

Heath-Brown, D. R., and Xiannan Li. 2017. "Prime Values of $a^2 + p^4$." *Invent. Math.* 208 (2): 441–99. https://doi.org/10.1007/s00222-016-0694-0.

Heath-Brown, D. R., and B. Z. Moroz. 2002. "Primes Represented by Binary Cubic Forms." *Proc. London Math. Soc. (3)* 84 (2): 257–88. https://doi.org/10.1112/plms/84.2.257.

Merikoski, Jori. 2023. "On the Largest Prime Factor of $n^2 + 1$." *J. Eur. Math. Soc. (JEMS)* 25 (4): 1253–84. https://doi.org/10.4171/jems/1216.

Pascadi, Alexandru. 2026. "Large Sieve Inequalities for Exceptional Maass Forms and the Greatest Prime Factor of $n^2 + 1$." *Forum Math. Pi* 14: Paper No. e8. https://doi.org/10.1017/fmp.2026.10025.

Tao, Terence, and Tamar Ziegler. 2008. "The Primes Contain Arbitrarily Long Polynomial Progressions." *Acta Math.* 201 (2): 213–305.