

# On prime values and prime factors of polynomials

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In a recent breakthrough, Pascadi (Pascadi 2026) proved that the greatest prime factor of  $x^2 + 1$  exceeds  $x^{1.3}$  for infinitely many integers  $x$ . This is the strongest result to date toward the famous fourth problem of Landau, which predicts that the polynomial  $x^2 + 1$  takes prime values infinitely often.

Many of the deepest achievements in modern analytic number theory can be viewed as instances of the following overarching problem.

**Question 1** *Let  $P_i(x_1, \dots, x_n)$ ,  $i = 1, \dots, k$ , be polynomials in  $n$  variables with integer coefficients. Under what conditions on the  $P_i$  are there infinitely many integer tuples  $(x_1, \dots, x_n)$  such that all the values*

$$P_1(x_1, \dots, x_n), \dots, P_k(x_1, \dots, x_n)$$

*are simultaneously prime?*

This formulation encompasses a remarkable range of classical and modern results.

The story begins in 1837 with Dirichlet's theorem on arithmetic progressions: if  $a$  and  $b$  are coprime integers, then the linear polynomial  $ax + b$  takes prime values for infinitely many integers  $x$ . This completely resolves Question 1 in the case of a single linear polynomial in one variable. It already illustrates a key principle: a simple necessary condition (here,  $\gcd(a, b) = 1$ ) is also sufficient.

The next layer of complexity appears with quadratic polynomials. Fermat asserted—and Euler proved—that every prime  $p \equiv 1 \pmod{4}$  can be written as

$$p = x^2 + y^2$$

for suitable integers  $x, y$ . Combined with Dirichlet's theorem, this shows that the quadratic form  $x^2 + y^2$  represents infinitely many primes.

Euler also established that there are infinitely many primes of the form  $x^2 + 2y^2$  and  $x^2 + 3y^2$ . Later, Dirichlet settled the case  $k = 1$  of Question 1 for binary quadratic forms

$$P(x, y) = ax^2 + bxy + cy^2,$$

and Iwaniec extended this to all irreducible quadratic polynomials in two variables that genuinely depend on both variables.

For polynomials of degree at least 3, progress proved much more difficult. A landmark result came in 1998, when Friedlander and Iwaniec (Friedlander and Iwaniec 1998) showed that the quartic form

$$x^2 + y^4$$

takes prime values infinitely often. This was the first example of a genuinely nonlinear, higher-degree polynomial in two variables known to produce infinitely many primes.

Shortly thereafter, Heath-Brown (Heath-Brown 2001) established the same for

$$x^3 + 2y^3,$$

and Heath-Brown and Moroz (Heath-Brown and Moroz 2002) extended the method to general irreducible cubic forms

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

with  $\gcd(a, b, c, d) = 1$ , provided that the form is not identically even. These works introduced powerful refinements of the Hardy–Littlewood circle method and sieve techniques.

In 2017, Heath-Brown and Li (Heath-Brown and Li 2017) proved that there are infinitely many primes of the form

$$x^2 + p^4,$$

where  $x$  is an integer and  $p$  is prime. This answers Question 1 for the pair of polynomials

$$P_1(x, y) = x^2 + y^4, \quad P_2(x, y) = y,$$

requiring both  $P_1$  and  $P_2$  to be prime.

A dramatic advance occurred in 2025, when Green and Sawhney (Green and Sawhney 2024) proved the first result in which *both* variables are required to be prime.

**Theorem 1** *If  $n \equiv 0$  or  $4 \pmod{6}$ , then there exist infinitely many primes of the form*

$$x^2 + ny^2$$

*with both  $x$  and  $y$  prime.*

They further observed that the same method should apply to general positive definite binary quadratic forms  $ax^2 + bxy + cy^2$  with  $b^2 - 4ac < 0$ , provided that there is no fixed prime divisor of  $xy(ax^2 + bxy + cy^2)$ . This effectively settles Question 1 for three polynomials

$$P_1 = ax^2 + bxy + cy^2, \quad P_2 = x, \quad P_3 = y$$

under the stated hypotheses.

Question 1 also encompasses longer patterns of primes. In 2008, Green and Tao (Green and Tao 2008) proved that for every  $k \geq 1$  there are infinitely many arithmetic progressions of length  $k$  consisting entirely of primes. This settles Question 1 for

$$P_i(x, y) = x + (i - 1)y, \quad i = 1, \dots, k,$$

with  $y \neq 0$ .

In the same year, Tao and Ziegler (Tao and Ziegler 2008) extended this to polynomial progressions

$$x + P_1(m), \dots, x + P_k(m),$$

for arbitrary integer polynomials  $P_i$  satisfying  $P_i(0) = 0$ . These results required a profound synthesis of additive combinatorics, ergodic theory, and analytic number theory.

Despite this extraordinary progress, many elementary-looking instances of Question 1 remain unresolved.

- The case  $P_1 = x, P_2 = x + 2$  is the twin prime conjecture.
- The case  $P_1 = x, P_2 = 2x + 1$  is the conjecture on Sophie Germain primes.
- Even the single-polynomial case  $k = 1, n = 1$  is open for every nonlinear polynomial  $P(x)$ .

A particularly famous example is Landau's fourth problem, which asks whether

$$x^2 + 1$$

is prime for infinitely many integers  $x$ .

Although we cannot yet prove that  $x^2 + 1$  is prime infinitely often, we can study its factorization. In 2010, Friedlander and Iwaniec (Friedlander and Iwaniec 2010, Theorem 25.9) showed that  $x^2 + 1$  is the product of two primes for infinitely many  $x$ . Obviously, the larger of two prime factors must exceed  $x$ .

In fact, much earlier, Chebyshev (with the first published proof due to Markov in 1895) proved that for every constant  $c > 0$ , the greatest prime factor of  $x^2 + 1$  exceeds  $cx$  for infinitely many  $x$ . Over more than a century, this linear bound was steadily improved.

Before 2026, the strongest result was due to Merikoski (Merikoski 2023), who proved that the greatest prime factor exceeds  $x^{1.279}$  infinitely often. Pascadi (Pascadi 2026) has now pushed this further.

**Theorem 2** *For infinitely many integers  $x$ , the greatest prime factor of  $x^2 + 1$  exceeds  $x^{1.3}$ .*

While this still falls short of proving that  $x^2 + 1$  is itself prime infinitely often, it represents the most substantial quantitative progress so far toward Landau's conjecture.

## References

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