A polynomial bound on Reidemeister moves for any fixed link

Bogdan Grechuk · 17 Feb 2026

A few days ago Marc Lackenby posted a preprint (Lackenby 2026) with a striking new result about links in $\mathbb{R}^3$. He proves that for any fixed link there is a polynomial bound controlling how complicated it is to pass between different diagrams of that link. More precisely, for each link K there exists a polynomial P_K such that if two diagrams of K have n_1 and n_2 crossings, then one can be transformed into the other using at most

$$P_K(n_1) + P_K(n_2)$$

Reidemeister moves.

This has an important computational consequence: for any fixed link K, the problem of deciding whether a given diagram represents K lies in the complexity class NP. In particular, there is a deterministic algorithm that solves this recognition problem in exponential time.

Let us recall some background. A *knot* is a closed, non-self-intersecting curve embedded in $\mathbb{R}^3$. More generally, a *link* $L \subset \mathbb{R}^3$ is a collection of $m \geq 1$ disjoint closed, non-self-intersecting curves. Thus knots are precisely the links with one component.

Two links L_1 and L_2 are said to be *equivalent* if one can be continuously deformed into the other without creating self-intersections. Formally, this means that there exists a continuous map

$$H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

such that for each t, the map $H_t = H(\cdot, t)$ is a homeomorphism, H_0 is the identity, and $H_1(L_1) = L_2$. This notion of equivalence preserves the number of components. In particular, knots are equivalent only to knots. A knot that is equivalent to a round circle is called the *unknot*.

In practice, knots and links are studied via planar diagrams. A *link diagram* is a projection of a link to the plane that is injective except at finitely many transverse double points, together with over/under information at each crossing. A diagram with no crossings is called *trivial*.

A foundational result, proved by Reidemeister in 1927, states that two diagrams represent equivalent links if and only if one can be transformed into the other by a finite sequence of three local moves, now known as *Reidemeister moves*. These consist of:

- twisting or untwisting a strand (Type I),
- sliding one strand over another (Type II),
- sliding a strand across a crossing (Type III).

Thus, a diagram represents the unknot if and only if it can be reduced to the trivial diagram by a sequence of Reidemeister moves. This naturally leads to a quantitative question: how many moves are needed?

In 2001, Hass and Lagarias (Hass and Lagarias 2001) proved that there exists a constant $c > 0$ such that any diagram of the unknot with n crossings can be simplified to the trivial diagram using at most 2^{cn} Reidemeister moves. An explicit estimate gives $c = 10^{11}$. While this was the first general upper bound, it grows exponentially in n.

A major breakthrough came in 2015, when Lackenby (Lackenby 2015) proved that unknot diagrams can in fact be simplified in polynomially many moves. Specifically, for every diagram of the unknot with n crossings, there exists a sequence of at most $(236n)^{11}$ Reidemeister moves that transforms it into the trivial diagram. Moreover, every intermediate diagram in the sequence has at most $(7n)^2$ crossings.

Lackenby's 2026 preprint extends the polynomial bound far beyond the unknot case.

Theorem 1 *For each equivalence class K of links in $\mathbb{R}^3$, there exists a polynomial P_K such that if D_1 and D_2 are two diagrams representing K with n_1 and n_2 crossings, then D_1 and D_2 can be transformed into each other by at most $P_K(n_1) + P_K(n_2)$ Reidemeister moves. Moreover, there is an algorithm that, given a diagram of a link in K, outputs the polynomial P_K.*

As a consequence, for any fixed link K, the problem of deciding whether a diagram with n crossings represents K lies in NP. This implies that there exist constants c and k (depending on K) such that recognition can be carried out in time at most $\exp(cn^k)$.

In the special case where K is the unknot, one may take $P_K(n) = (236n)^{11}$. More generally, the theorem guarantees that P_K can be made explicit in all concrete examples. For instance, for a fixed (p, q)–torus knot one obtains an explicit (though very large) polynomial bound independent of p and q (Lackenby 2026). A major remaining open question is whether there exists a universal polynomial $P(n)$ that works for every link K. If so, this would imply that the link equivalence problem (given two link diagrams, determine whether they represent equivalent links) lies in the complexity class NP and can be solved in exponential time. Currently, this fundamental problem is known to be solvable in finite time, but the number of steps in all known algorithms grows astronomically fast in the number of crossings.

References

Hass, Joel, and Jeffrey Lagarias. 2001. “The Number of Reidemeister Moves Needed for Unknotting.” *J. Amer. Math. Soc.* 14 (2): 399–428.

Lackenby, M. 2015. “A Polynomial Upper Bound on Reidemeister Moves.” *Ann. of Math.* 182 (2).

Lackenby, Marc. 2026. “A Polynomial Upper Bound on Reidemeister Moves for Each Link Type.” In *arXiv Preprint arXiv:2602.09923*. https://arxiv.org/abs/2602.09923.