The intersection exponent for planar random walks

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This post continues my series on favorite theorems of the 21st century. For an overview of the categories and my previous selections, see this earlier post. In the category *Between the Centuries*—that is, theorems proved in the late 20th century but published in the early 21st century—my favorite result in *Probability and Statistics* is the theorem of Lawler, Schramm, and Werner (Lawler et al. 2001), which establishes the exact value of the intersection exponent for two simple random walks in the plane.

Let $X_1, X_2, \dots$ be a sequence of independent, identically distributed vector-valued random variables, each taking values $(1,0)$, $(0,1)$, $(-1,0)$, or $(0,-1)$ with equal probability. The process

$$S_n = \sum_{k=1}^{n} X_k, \qquad n = 1, 2, \dots,$$

is called a *simple random walk* on $\mathbb{Z}^2$. Despite its elementary definition, this process exhibits remarkably subtle behavior, and several of its most natural quantitative properties resisted analysis for decades.

As an illustration, Erdős and Taylor asked in 1960 how often a planar random walk revisits its most frequently visited site during the first n steps. For each lattice point $(x, y) \in \mathbb{Z}^2$ and integer $n \geq 1$, let

$$T(n, x, y) = \#\{1 \leq k \leq n : S_k = (x, y)\}$$

be the number of visits to (x, y) up to time n, and define

$$T^*(n) = \max_{x,y} T(n, x, y).$$

Erdős and Taylor proved that, if the limit

$$\lim_{n \to \infty} \frac{T^*(n)}{(\log n)^2}$$

exists, then it must lie between $\frac{1}{4\pi}$ and $\frac{1}{\pi}$, and they conjectured that it exists almost surely and equals $\frac{1}{\pi}$. This conjecture remained open for more than forty years, until it was resolved in 2001 by Dembo, Peres, Rosen, and Zeitouni

(Dembo et al. 2001). Their approach first established an analogous result for two-dimensional Brownian motion, confirming a conjecture of Perkins and Taylor, and then transferred it to the discrete setting using invariance principles.

Another fundamental question concerns the likelihood that two independent planar random walks avoid each other. For a walk S and integer $k \geq 0$, write $S[0,k]$ for the (random) set of vertices visited up to time k. If the probability

$$P\big[S[0,k] \cap S'[0,k] = \emptyset\big]$$

decays asymptotically like $k^{-\gamma}$ for some $\gamma > 0$, then γ is called the *intersection exponent*. In 1988, Duplantier and Kwon, using methods from theoretical physics and conformal field theory, predicted that for two planar simple random walks one should have $\gamma = \frac{5}{8}$. A rigorous proof of this prediction was finally given in 2001 by Lawler, Schramm, and Werner (Lawler et al. 2001), using the then-new theory of Schramm–Loewner evolution (SLE).

Theorem 1 (Lawler–Schramm–Werner) *There exists a constant $c > 0$ such that*

$$c^{-1}k^{-5/8} \leq P\big[S[0,k] \cap S'[0,k] = \emptyset\big] \leq c\,k^{-5/8}$$

for all $k \geq 1$, where S and S' are two independent simple random walks starting from neighboring vertices in $\mathbb{Z}^2$.

A key input in the proof is an earlier result of Lawler and Puckette (Lawler and Puckette 2000), which shows that the discrete non-intersection probability above is, up to multiplicative constants, comparable to the corresponding probability for two planar Brownian motions. Lawler, Schramm, and Werner then computed the intersection exponents for n independent planar Brownian motions, showing that

$$\gamma_n = \frac{1}{24}\big(4n^2 - 1\big).$$

Specializing to $n = 2$ yields $\gamma_2 = \frac{5}{8}$, confirming the physicists' prediction and providing one of the earliest striking successes of SLE in probability theory.

References

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