Avi Wigderson's talk on Expander graphs

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This is a continuation of a series of posts about online seminars devoted to accessible presentations of some of the most significant mathematical theorems of the 21st century. I am happy to report that last Wednesday Prof. Boaz Klartag delivered a beautiful and accessible talk on sphere packing. If you missed the talk, its recording is now available on YouTube.

The next talk in this seminar series will take place on **Wednesday, February 11, 2026, at 4:00 PM (UK time)**. **Prof. Avi Wigderson** (Institute for Advanced Study, Princeton, USA) will speak on the topic of *expander graphs*.

To join the talk, simply click here at the scheduled time. No registration is required. For a full list of upcoming seminars, please visit the seminar webpage: https://th21.le.ac.uk/next-talks/.

For announcements of other talks, as well as descriptions of recent major mathematical breakthroughs, please see the full list of my posts on this blog.

Informally, expander graphs are simultaneously sparse and highly connected. This seemingly paradoxical combination of properties makes them extremely useful in a wide range of applications across mathematics, computer science, and even physics.

Let G be a graph with vertex set V. For a subset $S \subset V$, let $|\partial S|$ denote the number of edges with exactly one endpoint in S. The *edge expansion ratio* of G, denoted by $h(G)$, is defined by

$$h(G) = \min_{1 \le |S| \le |V|/2} \frac{|\partial S|}{|S|}. \quad (1)$$

A sequence of d-regular graphs $G_1, G_2, \dots$ of increasing size is called a *family of expanders* if there exists a constant $\epsilon > 0$ such that $h(G_i) \ge \epsilon$ for all i.

This notion admits an equivalent formulation in terms of eigenvalues of the adjacency matrix. The *adjacency matrix* $A = A(G)$ of an n-vertex graph G, whose vertices are labeled $1, 2, \dots, n$, is the $n \times n$ matrix with entries $a_{ij} = 1$ if vertices i and j are adjacent and $a_{ij} = 0$ otherwise. The matrix A has n real eigenvalues, which we order as

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G).$$

If G is d-regular, then $\lambda_1(G) = d$ and $\lambda_n(G) \geq -d$. In 1984, Dodziuk showed that for any d-regular graph G,

$$\frac{d - \lambda_2(G)}{2} \leq h(G) \leq \sqrt{2d\big(d - \lambda_2(G)\big)}.$$

In particular, a sequence of d-regular graphs forms a family of expanders if and only if there exists $\delta > 0$ such that $\lambda_2(G_i) \leq d - \delta$ for all i. Thus, the smaller $\lambda_2(G)$ is, the better an expander the graph G becomes.

The *Alon–Boppana inequality* asserts that for any d-regular n-vertex graph G,

$$\lambda_2(G) \geq 2\sqrt{d-1} - o_n(1), \tag{2}$$

where $o_n(1)$ denotes a quantity tending to 0 as $n \to \infty$ for fixed d. In 1986, Alon conjectured that for every $\epsilon > 0$, "almost all" d-regular n-vertex graphs G satisfy the reverse bound

$$\lambda_2(G) \leq 2\sqrt{d-1} + \epsilon. \tag{3}$$

In other words, almost all d-regular graphs are asymptotically optimal expanders.

This conjecture was proved by Friedman in 2008 (Friedman 2008). For even $d \geq 4$, a *random d-regular graph* on n vertices may be constructed from $d/2$ independent uniformly random permutations of $\{1, \ldots, n\}$.

Theorem 1 *Fix d and $\epsilon > 0$. Let p_n be the probability that a random d-regular graph $G_{d,n}$ satisfies (3). Then*

$$\lim_{n \to \infty} p_n = 1.$$

Although the statement above addresses even d, Friedman also established the conjecture for several other random graph models, including the uniform model on all d-regular graphs, thereby resolving the problem completely.

Theorem 1 shows that random regular graphs are expanders with overwhelming probability. However, this result does not provide an explicit construction, and explicit expanders are crucial in many theoretical and practical applications.

One classical source of explicit expanders comes from group theory: Cayley graphs of suitable groups often have strong expansion properties. A strikingly different, purely combinatorial construction was introduced by Reingold, Vadhan, and Wigderson (Reingold et al. 2002) via the *zig–zag product*.

For a graph G, a 2-*edge path* is a triple of vertices i, k, j such that $\{i, k\}$ and $\{k, j\}$ are edges of G. The graph G^2 has the same vertex set as G, with an edge between i and j whenever there exists a 2-edge path from i to j in G.

Let $[n] = \{1, 2, \ldots, n\}$. A d-regular graph G is called *d-coloured* if each edge is labeled by an element of $[d]$ in such a way that no two edges sharing a vertex have the same label. For a vertex v and a label $i \in [d]$, let $v[i]$ denote the neighbor of v along the edge labeled i.

Let G be a d_1-regular d_1-coloured graph on $[n_1]$, and let H be a d_2-regular d_2-coloured graph on $[d_1]$. The *zig–zag product* $G \cdot_Z H$ is the $(d_2)^2$-regular graph on $[n_1] \times [d_1]$ in which, for all $v \in [n_1]$, $k \in [d_1]$, and $i, j \in [d_2]$, there is an edge connecting

$$(v, k) \quad \text{to} \quad \big(v[k[i]],\, k[i][j]\big).$$

Theorem 2 *Let H be a d-regular graph on d^4 vertices with sufficiently large edge expansion. Define a sequence by $G_1 = H^2$ and $G_{i+1} = G_i^2 \cdot_Z H$ for $i \geq 1$. Then (G_i) is a family of d^2-regular expander graphs.*

Explicit expander constructions such as these play a fundamental role in modern theoretical computer science. Many randomized algorithms for combinatorial problems can be derandomized by replacing random graphs with carefully constructed expanders, leading to efficient deterministic algorithms with strong performance guarantees.

References

Friedman, Joel. 2008. *A Proof of Alon's Second Eigenvalue Conjecture and Related Problems*. Vol. 195. *Memoirs of the American Mathematical Society*. American Mathematical Soc.

Reingold, Omer, Salil Vadhan, and Avi Wigderson. 2002. "Entropy Waves, the Zig-Zag Graph Product, and New Constant-Degree Expanders." *Ann. Of Math.* 155 (1): 157–87.