Short-Interval Pseudorandomness of the Möbius Function

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In a paper that has just been accepted to Inventiones Mathematicae (Matomäki et al. 2026), Matomäki, Radziwiłł, Shao, Tao, and Teräväinen proved that famous Möbius function behaves like a random function on almost all intervals of the form $(x, x+H)$ for H as low as about cubic root of x.

One of the central themes of modern number theory is the idea that arithmetic functions often behave like random noise — unless there is a clear structural reason for them not to. Among the most mysterious of these objects is the *Möbius function* $\mu(n)$, defined for positive integers n by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes}, \\ 0 & \text{if } n \text{ is divisible by a square.} \end{cases} \tag{1}$$

At a heuristic level, $\mu(n)$ flips signs unpredictably and frequently vanishes. This has led to the guiding philosophy that $\mu(n)$ should exhibit strong *pseudorandomness*: it should have very little correlation with structured or deterministic sequences.

Understanding this randomness is not merely a philosophical pursuit — it has deep consequences for prime numbers, additive patterns, and long-standing conjectures such as the Riemann Hypothesis.

A natural way to test pseudorandomness is to measure how much $\mu(n)$ correlates with oscillatory functions such as exponentials

$$e(t) = e^{2\pi i t}.$$

If $\mu(n)$ behaves randomly, then sums like

$$\sum_{n \le x} \mu(n) e(-\alpha n)$$

should exhibit strong cancellation for every real frequency α.

In a landmark 1991 result, Zhan (Zhan 1991) proved that this is indeed the case even when the sum is restricted to *short intervals*:

$$\sup_{\alpha \in \mathbb{R}} \frac{1}{H} \left| \sum_{x \le n \le x+H} \mu(n) e(-\alpha n) \right| \le C \log^{-A}(x),$$

whenever

$$x^{5/8+\varepsilon} \le H \le x.$$

In words: over intervals as short as $x^{5/8+\varepsilon}$, the Möbius function has essentially no correlation with linear exponential phases.

A major conceptual advance came in 2023 through work of Matomäki, Shao, Tao, and Teräväinen (Matomäki et al. 2023). They showed that the same phenomenon holds not only for linear phases αn, but for *any polynomial phase*.

Let $\mathcal{P}_k$ denote the set of real polynomials of degree at most k.

Theorem 1 *Let $x \ge 3$ and suppose $x^{5/8+\varepsilon} \le H \le x^{1-\varepsilon}$. Then for any $A > 0$,*

$$\sup_{P \in \mathcal{P}_k} \frac{1}{H} \left| \sum_{x \le n \le x+H} \mu(n) e(-P(n)) \right| \le C \log^{-A}(x),$$

where C depends only on k, A, and ε.

This result says that the Möbius function does not correlate with any low-complexity deterministic pattern — even locally.

A natural question that remains is *how short can these intervals be while still exhibiting randomness?*

Now Matomäki, Radziwiłł, Shao, Tao, and Teräväinen (Matomäki et al. 2026) made a major breakthrough by lowering the exponent from $5/8$ to $1/3$ — nearly the theoretical limit for current techniques — provided that we allow a small exceptional set.

Theorem 2 *Let $X \ge 3$ and suppose $X^{1/3+\varepsilon} \le H \le X^{1-\varepsilon}$ for some $\varepsilon > 0$. Let $d \ge 1$, and for each $x \in [X, 2X]$ let $P_x : \mathbb{Z} \to \mathbb{R}$ be a polynomial of degree d. Let $A > 0$. Then inequality*

$$\frac{1}{H} \left| \sum_{x \le n \le x+H} \mu(n) e(-P_x(n)) \right| \le \log^{-A}(X)$$

holds for all $x \in [X, 2X]$ outside of a set of measure $O_{A,d,\varepsilon}\left(X \log^{-A} X\right)$.

Thus, Möbius randomness persists on intervals nearly as short as $X^{1/3}$ for almost every location.

These results provide striking quantitative evidence for a central belief of analytic number theory: *The Möbius function behaves like random noise*. Each improvement in interval length sharpens our understanding of how chaos and structure coexist in the integers.

Matomäki, Kaisa, Maksym Radziwiłł, Xuancheng Shao, Terence Tao, and Joni Teräväinen. 2026. "Higher Uniformity of Arithmetic Functions in Short Intervals II. Almost All Intervals." *Inventiones Mathematicae*. https://doi.org/10.1007/s00222-026-01408-6.

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Zhan, Tao. 1991. "On the Representation of Large Odd Integer as a Sum of Three Almost Equal Primes." *Acta Math. Sinica (N.S.)* 7 (3): 259–72. https://doi.org/10.1007/BF02583003.