On infinite-index subgroups of one-relator groups

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A recent paper by Henry Wilton (Wilton 2024), just accepted to *Acta Mathematica*, settles a long-standing question regarding the structure of one-relator groups. Wilton proves that free groups and surface groups are the only examples of one-relator groups in which every subgroup of infinite index is free.

This result provides a definitive classification in a field that bridges algebra and topology. To appreciate the significance of this theorem, let us first revisit the foundational concepts of group presentations and their topological origins.

Recall that a group is a pair $(G, \cdot)$ where the binary operation satisfies associativity, possesses an identity element, and admits inverses. While the integers $(\mathbb{Z}, +)$ provide a simple commutative example, geometric group theory often deals with non-commutative groups defined by symbolic sequences.

Let S be a set of symbols called *generators*. For each $x \in S$, we introduce a formal inverse x^{-1}. A *word* is any finite sequence of these symbols. We consider the words xx^{-1} and $x^{-1}x$ to be "trivial".

Two words are deemed equivalent if one can be transformed into the other by inserting or deleting trivial words. The set of equivalence classes forms a group under concatenation, denoted by F_S, and is called the *free group* on S. If S is finite, F_S is called a free group of finite rank. For instance, the integers $\mathbb{Z}$ are isomorphic to the free group on a single generator.

Most groups are not free; they satisfy constraints. Let $R \subset F_S$ be a set of words. We say that words $w_1 \in F_S$ and $w_2 \in F_S$ are R*-equivalent* if they can be transformed into each other by insertions or deletions either of trivial words or words from the set R. The set of equivalence classes of words under R-equivalence form a group with the same operation $w_1 \cdot w_2 = w_1 w_2$ which we denote by $\langle S|R \rangle$. If a group G is isomorphic to a group $\langle S|R \rangle$ for some set S and some $R \subset F_S$, we will call $\langle S|R \rangle$ a *presentation* of G, where S is called the set of *generators*, and R is called the set of *relators*. A group G is called *finitely presented* if it has a presentation $\langle S|R \rangle$ in which both S and R are finite sets.

- If $R = \emptyset$, the group is free.

- If $|R| \leq 1$ (the set R contains at most one element), the group $G = \langle S \mid R \rangle$ is called a *one-relator group*.

One-relator groups appear naturally in topology. Consider a path-connected k-manifold M. Fix a basepoint $x_0 \in M$ and consider the set of loops starting and ending at x_0. Let us call two such loops equivalent if they can be continuously transformed into each other within M. The union of two loops is defined in an obvious way: travel along the first loop, then along the second. Then the set of equivalence classes of such loops with the union operation forms a group. This group, up to isomorphism, depends only on M but not on x_0, is denoted $\pi_1(M)$, and is called the the fundamental group of M. It has been introduced by Poincaré in 1895, and since then became a central object of study in topology.

- If M is simply connected (e.g., a sphere), the fundamental group is trivial.
- If M is a closed surface other than the 2-sphere, $\pi_1(M)$ is called a *surface group*.

Crucially, all surface groups are finitely presented and, in fact, are **one-relator groups**. This connects the algebraic definition directly to the topology of surfaces.

The motivation for Wilton's theorem lies in the subgroup structure of the mentioned groups.

- **Free Groups:** The famous *Nielsen-Schreier Theorem* states that every subgroup of a free group is free.
- **Surface Groups:** While not all subgroups of a surface group are free, a standard result in topology states that any subgroup of *infinite index* must be free.

Recall that a (left) coset of a subgroup H of group G with respect to $g \in G$ is the set $\{gh : h \in H\}$. We say that subgroup H is of *infinite index* if there are infinitely many different cosets of H in G.

A major question in geometric group theory has been whether free groups and surface groups are the *only* finitely presented groups with this specific subgroup property. Historically, this question received significant attention in the context of one-relator groups, where the answer was conjectured to be "Yes."

Culminating a long chain of partial results, Wilton (Wilton 2024) has confirmed this conjecture:

Theorem 1 *Let G be an infinite one-relator group. If every subgroup of infinite index in G is free, then G is isomorphic to either a free group or a surface group.*

Stated contrapositively: Unless G is free or a surface group, G must contain a subgroup of infinite index that is not free. This result serves as a powerful classification tool and a useful precursor to the broader problem of identifying surface subgroups within larger structures.

Wilton, Henry. 2024. "Surface Groups Among Cubulated Hyperbolic and One-Relator Groups." *arXiv Preprint arXiv:2406.02121*. https://arxiv.org/abs/2406.02121.