All braid groups are linear

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This post continues my series on favorite theorems of the 21st century. For an overview of the categories and my previous selections, see this earlier post. In the category *Between the Centuries* in Algebra, my favorite theorem is the linearity of braid groups, established independently by Krammer (Krammer 2002) and Bigelow (Bigelow 2001).

The set of all invertible $n \times n$ matrices with entries in a field F forms a group under matrix multiplication. This group is denoted by $\mathrm{GL}_n(F)$ and is called the *general linear group* of degree n over F. A group G is said to be *linear* if it is isomorphic to a subgroup of $\mathrm{GL}_n(F)$ for some natural number n and some field F. Linearity is a powerful property: linear groups enjoy a rich supply of algebraic and geometric tools, and showing that a group is linear grants access to these tools "for free." Nevertheless, proving linearity can be highly nontrivial. For a long time, braid groups were a prominent example for which this question remained open.

We now briefly recall the definition of braid groups. Let X_1 be a set of m points in the plane with coordinates

$$(x, 1), (x, 2), \dots, (x, m),$$

and let X_2 be a parallel set of points with coordinates

$$(y, 1), (y, 2), \dots, (y, m),$$

where $y > x$. Consider m (not necessarily straight) strands connecting points of X_1 to points of X_2, such that each point is the endpoint of exactly one strand. The strands are required to run from left to right. Whenever two strands intersect, one is designated as passing over the other, and this over–under information is fixed. Such a configuration is called a *braid*. The precise geometric shape of the strands, as well as the specific values of x and y, are irrelevant: two configurations represent the same braid if one can be continuously deformed into the other without breaking or merging strands. What matters is the pattern of intertwining.

Now let X be a braid from X_1 to X_2, and let Y be another braid from X_2 to a third set of points

$$X_3 = \{(z, 1), (z, 2), \dots, (z, m)\}$$

with $z > y > x$. By placing Y to the right of X and concatenating the strands, we obtain a new braid Z from X_1 to X_3. This operation defines the product of braids, written $Z = X \times Y$. The set B_m of all braids with m strands forms a group under this operation, called the *braid group* on m strands.

In 1935, Burau proposed a representation of braid groups B_m by matrices over a field, now known as the Burau representation. However, he did not determine whether this representation is faithful, that is, whether distinct braids are always sent to distinct matrices. In 1991, Moody showed that the Burau representation fails to be faithful in general. The problem of linearity of braid groups was finally resolved when Krammer (Krammer 2002) and, independently, Bigelow (Bigelow 2001) proved that a different representation—introduced earlier by Krammer (Krammer 2000)—is faithful. As a consequence, all braid groups are linear.

Theorem 1 *For every m, there exist a natural number n, a field K, and a map ρ assigning to any braid X with m strands an $n \times n$ matrix $\rho(X)$ over the field K, such that*

(a) $\rho(X \times Y) = \rho(X) \cdot \rho(Y)$ for any two braids X,Y, and

(b) $\rho(X) \neq \rho(Y)$ whenever $X \neq Y$ (different matrices correspond to different braids).

In other words, all braid groups are linear.

References

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