On the solubility of the negative Pell equation

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The list of accepted papers to *Acta Mathematica* includes the recent work of Koymans and Pagano (Koymans and Pagano 2022), scheduled for publication in the first issue of 2026. This paper marks the culmination of several decades of research by confirming a celebrated conjecture of Peter Stevenhagen concerning the solvability of the *negative Pell equation*. The result provides a precise asymptotic density for those parameters for which this classical Diophantine equation admits solutions.

Let $d > 0$ be a square-free integer, and consider the problem of approximating the irrational number $\sqrt{d}$ by rational numbers. A rational number $\frac{x}{y}$ (with $y \neq 0$) is a good approximation to $\sqrt{d}$ if

$$\frac{x}{y} \approx \sqrt{d},$$

or equivalently,

$$x^2 \approx dy^2.$$

Since $x^2 - dy^2 = 0$ has no nontrivial integer solutions when d is not a perfect square, the "best" possible cases are $x^2 - dy^2 = m$, where $m = 1$ or $m = -1$. These are known respectively as the *positive Pell equation* and the *negative Pell equation*.

A classical result in number theory asserts that the positive Pell equation

$$x^2 - dy^2 = 1$$

has infinitely many integer solutions for every positive square-free integer d. These solutions arise from the periodic continued fraction expansion of $\sqrt{d}$ and provide increasingly accurate rational approximations of $\sqrt{d}$ from above.

The situation for the negative Pell equation

$$x^2 - dy^2 = -1 \tag{1}$$

is markedly different. If (1) has *one* integer solution, then it automatically has infinitely many, again yielding a sequence of rational approximations of $\sqrt{d}$—this time from below. However, unlike the positive case, the solvability is not guaranteed.

For instance, when $d = 3$, the equation $x^2 - 3y^2 = -1$ has no solutions modulo 3 and therefore no integer solutions. More generally, if d has a prime divisor p congruent to $3 \pmod 4$, then (1) has no solutions modulo p and is therefore unsolvable in integers. Let S denote the set of positive square-free integers with no prime divisors of the form $4k + 3$. Even within this restricted set, solvability is subtle: for example, $d = 34 \in S$, yet $x^2 - 34y^2 = -1$ has no integer solutions.

In 1993, Stevenhagen (Stevenhagen 1993) proposed a striking conjecture describing how often the negative Pell equation is solvable. To state it precisely, define:

- $f(N)$: the number of positive square-free integers $d \leq N$ for which (1) has an integer solution;
- $g(N)$: the number of elements of S in $\{1, \ldots, N\}$;
- $\alpha = \prod_{j=1}^{\infty}(1 + 2^{-j})^{-1} = 0.419\ldots$.

Stevenhagen conjectured that, asymptotically, a fixed proportion of values $d \in S$ admit solutions to the negative Pell equation, namely

$$\lim_{N\to\infty} \frac{f(N)}{g(N)} = 1 - \alpha = 0.580\ldots$$

The first major breakthrough came in 2010, when Fouvry and Klüners (Fouvry and Klüners 2010a) proved that there exist constants $0 < C_1 \leq C_2 < 1$ such that

$$(C_1 - o(1))g(N) \leq f(N) \leq (C_2 + o(1))g(N), \qquad \text{as } N \to \infty.$$

More precisely, they showed that one can take $C_1 = \alpha$ and $C_2 = \frac{2}{3}$, thereby establishing nontrivial lower and upper density bounds.

Later the same year, Fouvry and Klüners (Fouvry and Klüners 2010b) refined their methods to improve the lower bound to

$$C_1 = \frac{5\alpha}{4} = 0.524\ldots$$

Subsequent advances continued to narrow the gap. In 2022, Chan, Koymans, Milovic, and Pagano (Chan et al. 2022) pushed the lower bound further to approximately 0.538. Shortly thereafter, Koymans and Pagano (Koymans and Pagano 2020) improved both bounds simultaneously, obtaining

$$C_1 = 0.543\ldots \quad \text{and} \quad C_2 = 0.599\ldots$$

Finally, in their recent work (Koymans and Pagano 2022), Koymans and Pagano closed the remaining gap by proving that the lower and upper bounds coincide. This result confirms Stevenhagen's conjecture in full.

Theorem 1 (Koymans–Pagano)

$$\lim_{N\to\infty} \frac{f(N)}{g(N)} = 1 - \prod_{j=1}^{\infty}(1+2^{-j})^{-1} = 0.580\ldots$$

This theorem represents a definitive answer to a classical problem at the intersection of Diophantine equations, algebraic number theory, and arithmetic statistics. It also highlights the remarkable predictive power of heuristic models in modern number theory.

References

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