The hottest and coldest points on a triangle

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In a paper recently accepted to *Inventiones Mathematicae* (Chen, Gui, and Yao 2026), Chen, Gui, and Yao resolved a natural and long-standing refinement of the hot spots problem for triangles. They proved that if a triangular piece of metal is given an initial heat distribution and the heat is allowed to diffuse freely, then after a sufficiently long time the hottest and coldest points occur *precisely* at the endpoints of the longest side of the triangle.

The celebrated *hot spots conjecture* predicts, in non-technical terms, that for a flat piece of insulated metal the eventual hottest and coldest points lie on the boundary of the domain. To state the conjecture rigorously, some definitions are required.

Let $\Omega \subset \mathbb{R}^2$ be a *Lipschitz domain*, that is, a bounded domain whose boundary $\partial\Omega$ has the property that for every $x \in \partial\Omega$ there exists a neighbourhood U such that $\partial\Omega \cap U$ can be represented as the graph of a Lipschitz function in some orthonormal coordinate system.

The *second Neumann eigenvalue* $\mu_2 = \mu_2(\Omega)$ is the smallest positive real number for which there exists a nontrivial smooth function $u : \Omega \to \mathbb{R}$ satisfying

$$\Delta u = -\mu_2 \, u \quad \text{in } \Omega,$$

together with the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

at all smooth points of $\partial\Omega$, where n denotes the outward unit normal vector. Any such function u is called a *second Neumann eigenfunction* of Ω.

Proposed by Rauch in 1975, the hot spots conjecture asserts that the extrema of the second Neumann eigenfunction are attained on the boundary $\partial\Omega$. In 1999, Burdzy and Werner (Burdzy and Werner 1999) constructed a counterexample in a highly non-convex planar domain with holes, showing that the conjecture does not hold in full generality. Nevertheless, it is widely believed to be true for simply connected domains, and in particular for convex ones.

A major advance was made in 2004 by Atar and Burdzy (Atar and Burdzy 2004), who proved the conjecture for a broad class of planar domains. Specifically, if $\Omega \subset \mathbb{R}^2$ is a bounded, open, connected Lipschitz domain of the form

$$\Omega = \{(x_1, x_2) : f_1(x_1) < x_2 < f_2(x_1)\},$$

where f_1 and f_2 are Lipschitz functions with Lipschitz constant 1, then every eigenfunction corresponding to $\mu_2(\Omega)$ attains its maximum and minimum only at boundary points.

Despite its strength, this result does not cover many elementary domains. One of the simplest nontrivial bounded domains is a triangle, yet even in this case the hot spots conjecture turned out to be surprisingly subtle. In 1999, Bañuelos and Burdzy proved the conjecture for obtuse triangles, while the Atar–Burdzy theorem implies it for right triangles. Acute triangles proved more challenging: in 2012 they became the subject of a Polymath project involving many contributors.

This line of work culminated in a result of Judge and Mondal (Judge and Mondal 2020), who in 2020 proved the hot spots conjecture for *all* triangles. That is, for any triangular domain Ω, the second Neumann eigenfunction attains its extrema on $\partial\Omega$. A gap in the original argument was later identified and corrected by the same authors in an erratum (Judge and Mondal 2022).

While this settled the conjecture for triangles, it left open a natural geometric question: *where exactly on the boundary do the extrema occur?* This question was finally answered by Chen, Gui, and Yao in (Chen, Gui, and Yao 2026).

Theorem 1 *For any triangle Ω, the second Neumann eigenfunction is unique (up to multiplication by a constant), and its global maximum and minimum are attained exactly at the endpoints of the longest side of Ω.*

This result provides a strikingly precise description of the long-time behavior of heat flow on triangular domains, completing the story of hot spots for triangles.

References

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