Squaring the circle with Jordan measurable pieces

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The January 2026 issue of *Advances in Mathematics* contains a striking paper by Máthé, Noel, and Pikhurko (Máthé et al. 2026), showing that a circle can be partitioned into finitely many *Jordan measurable* pieces which can then be rearranged, using only translations, to form a square of the same area. This result represents the latest milestone in a century-long story at the intersection of geometry, measure theory, and the foundations of mathematics.

The *axiom of choice* is one of the cornerstones of modern set theory. It asserts that given any collection S of pairwise disjoint nonempty sets, one can select exactly one element from each set. While this principle may appear self-evident, it has famously counterintuitive consequences. One of the most celebrated is the Banach–Tarski paradox: for any $d \geq 3$, any two bounded subsets of $\mathbb{R}^d$ with nonempty interiors are *equidecomposable*. That is, one can partition one set into finitely many pieces and, using only translations and rotations, reassemble them to obtain the other set. In particular, a single unit ball in $\mathbb{R}^3$ can be decomposed and rearranged to form two identical unit balls.

For decades, such decompositions necessarily relied on highly non-measurable pieces. A major breakthrough came in 2022, when Grabowski, Máthé, and Pikhurko (Grabowski et al. 2022) proved that if the sets involved are measurable and have the same measure, then the pieces can also be chosen to be measurable.

In contrast to the higher-dimensional situation, Tarski showed that the Banach–Tarski phenomenon cannot occur in the plane. If a measurable set $A \subset \mathbb{R}^2$ is partitioned into (possibly non-measurable) pieces and rearranged by translations and rotations to form another measurable set B, then A and B must have the same area. This led Tarski, in 1925, to ask a natural and deceptively simple question: *are any two planar sets of equal area equidecomposable in this way?* In particular, can a disk be rearranged to form a square of the same area? This became known as *Tarski's circle squaring problem*.

In 1990, Laczkovich (Laczkovich 1990) answered this question affirmatively. He showed that a disk can indeed be partitioned into finitely many pieces which can be rearranged—using translations alone, without rotations—to form a square. However, the pieces arising in his construction are extremely intricate and, crucially, non-measurable. Laczkovich therefore left open the question of whether such a decomposition could be achieved using measurable pieces.

This problem was resolved in 2017 by Grabowski, Máthé, and Pikhurko (Grabowski et al. 2017), who showed that a disk and a square of the same area are equidecomposable by translations *with Lebesgue measurable parts*. Formally, two sets $A, B \subset \mathbb{R}^2$ are said to be *equidecomposable by translations* if there exist partitions

$$A = A_1 \cup \cdots \cup A_m, \qquad B = B_1 \cup \cdots \cup B_m,$$

such that $B_i = A_i + v_i$ for some vectors $v_1, \ldots, v_m \in \mathbb{R}^2$. If, moreover, all A_i (and hence all B_i) are Lebesgue measurable, then the equidecomposition is said to be *with measurable parts*.

At this point, reactions to the solution of Tarski's problem begin to diverge. Some find the result astonishing; others remark that if the axiom of choice can turn one ball into two, then transforming a disk into a square seems comparatively modest. Indeed, the proof in (Grabowski et al. 2017) makes essential use of the axiom of choice, albeit restricted to a null set, and the measurable pieces involved are still far from simple. In particular, they are not *Borel* sets—those obtained from open sets by countable unions, intersections, and set differences.

Remarkably, also in 2017, Marks and Unger (Marks and Unger 2017) strengthened this result by proving that a disk and a square of the same area are equidecomposable by translations using *Borel* pieces. Even more strikingly, their proof avoids the axiom of choice altogether. Results of this kind are said to be *constructive*. The existence of such a constructive solution answered a question posed by Wagon in 1985.

Despite this progress, the pieces in the Marks–Unger construction remain quite complicated from a geometric perspective. In particular, they lack a natural regularity property known as *Jordan measurability*. A bounded set $X \subset \mathbb{R}^d$ is Jordan measurable if its boundary has Lebesgue measure zero. Intuitively, this means that X can be well approximated by a finite grid: when a square is subdivided into an $n \times n$ grid, the boundary of X intersects only a negligible fraction of the small squares, so almost all squares lie entirely inside or outside X. As a result, Jordan measurable sets admit finite approximations to arbitrary precision.

This brings us back to the 2026 work of Máthé, Noel, and Pikhurko (Máthé et al. 2026). They proved that a disk and a square of the same area are equidecomposable by translations in such a way that *every piece is Jordan measurable*, and at the same time Borel. Their result follows from a far-reaching general theorem: for any $d \geq 1$, if A and B are bounded subsets of $\mathbb{R}^d$ with equal positive measure and boundaries of upper Minkowski dimension strictly less than d, then A and B are equidecomposable by translations. Moreover, the pieces can be chosen so that their boundaries also have upper Minkowski dimension less than d, and if A and B are Borel, so are the pieces.

In a sense, this result achieves the best possible outcome. In 1963, Dubins, Hirsch, and Karush introduced the notion of *scissors congruence* in the plane, where equidecompositions are required to use pieces whose boundaries consist of a single Jordan curve. They showed that a disk is not scissors congruent to a square—or indeed to any convex set other than a disk itself. Thus, while perfect geometric simplicity is unattainable, the Jordan measurable pieces constructed in (Máthé et al. 2026) represent the "nicest" pieces one could reasonably hope for in the long quest to square the circle.

References

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