Exponentially better Ramsey upper bounds

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An exciting paper by Balister, Bollobás, Campos, Griffiths, Hurley, Morris, Sahasrabudhe, and Tiba (Balister et al. 2026) has just been accepted to the *Journal of the American Mathematical Society* (JAMS). In this work, the authors establish a new upper bound for Ramsey numbers with an arbitrary number l of colours. Remarkably, when $l \geq 3$, their bound is exponentially smaller than all previously known bounds, representing a major breakthrough in the area.

Let K_n denote the complete graph on n vertices. In his seminal 1930 paper, Ramsey proved that for any positive integers k and m, there exists an integer n such that every red–blue colouring of the edges of K_n contains either a red copy of K_k or a blue copy of K_m. The smallest such n is denoted by $r(k,m)$ and is called the *(two-colour) Ramsey number*. The special case $r(k,k)$ is referred to as the *diagonal Ramsey number*.

Since Ramsey's foundational result, estimating Ramsey numbers has become one of the central problems in graph theory and combinatorics. A first general upper bound was obtained in 1935 by Erdős and Szekeres, who proved that

$$r(k+1, m+1) \leq \binom{k+m}{k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient. In particular, this implies that

$$r(k+1, k+1) \leq \binom{2k}{k}. \tag{1}$$

Asymptotically, this bound grows like 4^k up to lower-order factors. For nearly 90 years, a long sequence of works succeeded in refining only these lower-order terms, while the leading exponential constant 4 remained unchanged.

A dramatic shift occurred with the recent work of Campos, Griffiths, Morris, and Sahasrabudhe (Campos et al. 2026), accepted for publication in the *Annals of Mathematics*. They obtained the first genuine exponential improvement over the Erdős–Szekeres bound.

Theorem 1 *There exists a constant $\epsilon > 0$ such that*

$$r(k,k) \leq (4-\epsilon)^k \tag{2}$$

for all sufficiently large integers k.

The authors showed that one may take $\epsilon = 2^{-7}$, and noted that further technical optimisation could improve this value. This was subsequently carried out by Gupta, Ndiaye, Norin, and Wei (Gupta et al. 2024), who proved Theorem 1 with $\epsilon = 0.2007\ldots$.

While Theorem 1 resolves a long-standing problem for the two-colour case, Ramsey theory naturally extends to multiple colours. For an integer $l \geq 2$, let $R(k, l)$ denote the smallest integer n such that every colouring of the edges of K_n with l colours contains a monochromatic copy of K_k. With this notation, $R(k, 2) = r(k, k)$. A major theme in the subject is to study $R(k, l)$ for fixed l as $k \to \infty$.

A straightforward generalisation of the Erdős–Szekeres argument yields the upper bound

$$R(k, l) \leq l^{lk+o(k)}. \tag{3}$$

For many decades, this remained the best known estimate, up to improvements in the lower-order $o(k)$ term. In their JAMS paper, Balister et al. (Balister et al. 2026) obtained the first exponential improvement over (3).

Theorem 2 *For each fixed integer $l \geq 2$, there exists a constant $\delta = \delta(l) > 0$ such that*

$$R(k, l) \leq e^{-\delta k}\, l^{lk}$$

for all sufficiently large integers k.

In the case $l = 2$, Theorem 2 provides an alternative—and significantly shorter —proof of Theorem 1. For all $l \geq 3$, Theorem 2 is new.

For comparison, the best known lower bound for multicolour Ramsey numbers has the form

$$R(k, l) \geq \left(2^{C(l-2)+1/2}\right)^{k+o(k)},$$

where $C \approx 0.383796$ (Sawin 2022). In particular, for $l = 2$ we only know that

$$r(k, k) = R(k, 2) \geq (\sqrt{2})^{k+o(k)}.$$

This remains vastly smaller than the current upper bounds in Theorems 1 and 2. Closing this exponential gap is one of the most intriguing open problems in Ramsey theory, and the true growth rate of Ramsey numbers continues to be a deep mystery.

References

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