Exponentially better Ramsey upper bounds

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An exciting paper by Balister, Bollobás, Campos, Griffiths, Hurley, Morris, Sahasrabudhe, and Tiba (Balister et al. 2026) has just been accepted to the *Journal of the American Mathematical Society* (JAMS). In this work, the authors establish a new upper bound for Ramsey numbers with an arbitrary number l of colours. Remarkably, when $l \geq 3$, their bound is exponentially smaller than all previously known bounds, representing a major breakthrough in the area.

Let K_n denote the complete graph on n vertices. In his seminal 1930 paper, Ramsey proved that for any positive integers k and m, there exists an integer n such that every red–blue colouring of the edges of K_n contains either a red copy of K_k or a blue copy of K_m. The smallest such n is denoted by $r(k,m)$ and is called the *(two-colour) Ramsey number*. The special case $r(k,k)$ is referred to as the *diagonal Ramsey number*.

Since Ramsey's foundational result, estimating Ramsey numbers has become one of the central problems in graph theory and combinatorics. A first general upper bound was obtained in 1935 by Erdős and Szekeres, who proved that

$$r(k+1, m+1) \leq \binom{k+m}{k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient. In particular, this implies that

$$r(k+1, k+1) \leq \binom{2k}{k}. \tag{1}$$

Asymptotically, this bound grows like 4^k up to lower-order factors. For nearly 90 years, a long sequence of works succeeded in refining only these lower-order terms, while the leading exponential constant 4 remained unchanged.

A dramatic shift occurred with the recent work of Campos, Griffiths, Morris, and Sahasrabudhe (Campos et al. 2026), accepted for publication in the *Annals of Mathematics*. They obtained the first genuine exponential improvement over the Erdős–Szekeres bound.

Theorem 1 *There exists a constant $\epsilon > 0$ such that*

$$r(k,k) \leq (4-\epsilon)^k \tag{2}$$

for all sufficiently large integers k.

The authors showed that one may take $\epsilon = 2^{-7}$, and noted that further technical optimisation could improve this value. This was subsequently carried out by Gupta, Ndiaye, Norin, and Wei (Gupta et al. 2024), who proved Theorem 1 with $\epsilon = 0.2007\ldots$.

While Theorem 1 resolves a long-standing problem for the two-colour case, Ramsey theory naturally extends to multiple colours. For an integer $l \geq 2$, let $R(k,l)$ denote the smallest integer n such that every colouring of the edges of K_n with l colours contains a monochromatic copy of K_k. With this notation, $R(k,2) = r(k,k)$. A major theme in the subject is to study $R(k,l)$ for fixed l as $k \to \infty$.

A straightforward generalisation of the Erdős–Szekeres argument yields the upper bound

$$R(k,l) \leq l^{lk+o(k)}. \tag{3}$$

For many decades, this remained the best known estimate, up to improvements in the lower-order $o(k)$ term. In their JAMS paper, Balister et al. (Balister et al. 2026) obtained the first exponential improvement over (3).

Theorem 2 *For each fixed integer $l \geq 2$, there exists a constant $\delta = \delta(l) > 0$ such that*

$$R(k,l) \leq e^{-\delta k}\, l^{lk}$$

for all sufficiently large integers k.

In the case $l = 2$, Theorem 2 provides an alternative—and significantly shorter—proof of Theorem 1. For all $l \geq 3$, Theorem 2 is new.

For comparison, the best known lower bound for multicolour Ramsey numbers has the form

$$R(k,l) \geq \left(2^{C(l-2)+1/2}\right)^{k+o(k)},$$

where $C \approx 0.383796$ (Sawin 2022). In particular, for $l = 2$ we only know that

$$r(k,k) = R(k,2) \geq (\sqrt{2})^{k+o(k)}.$$

This remains vastly smaller than the current upper bounds in Theorems 1 and 2. Closing this exponential gap is one of the most intriguing open problems in Ramsey theory, and the true growth rate of Ramsey numbers continues to be a deep mystery.

Update 13.04.2026 The paper (Balister et al. 2026) is now published. The reference is updated to the published version.

Update 05.05.2026 The paper (Campos et al. 2026) is now published. The reference is updated to the published version.

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