Cycle decompositions of complete graphs

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This post continues my series on favorite theorems of the 21st century. For a complete overview of the categories and list of my previous selections, see this previous blog post. In the category *Between the Centuries* within Combinatorics, my personal favorite is the cycle decomposition theorem of Šajna.

One of the oldest results in combinatorics is Kirkman's theorem from 1847, which states that for every integer

$$n \equiv 1 \text{ or } 3 \pmod 6,$$

the edges of the complete graph K_n can be partitioned into edge-disjoint triangles. It is straightforward to verify that these congruence conditions on n are necessary. Kirkman's theorem became the starting point of several long-running research programs, some of which spanned more than a century.

One such program asked for decompositions of K_n into cycles C_m of a fixed length $m \geq 3$. We say that a graph G is C_m*-decomposable* if its edge set can be written as a disjoint union of cycles of length m (these cycles need not be vertex-disjoint).

A particularly famous special case is $m = n$. A classical result, attributed to Walecki by Lucas in 1892, asserts that for every odd n the complete graph K_n can be decomposed into cycles of length n. Such cycles are called Hamiltonian cycles.

There are several obvious necessary conditions for a C_m-decomposition of K_n to exist. First, one must have $3 \leq m \leq n$. Second, the number of edges of K_n, namely $n(n-1)/2$, must be divisible by m. Finally, since every cycle contributes degree 2 at each of its vertices, all vertex degrees in K_n must be even, which forces n to be odd.

In a landmark paper from 2002, Šajna (Šajna 2002) proved that these obvious necessary conditions are, in fact, sufficient. Her result culminated a long sequence of partial advances and brought a century-old project to a definitive conclusion. She also established an analogous statement for even n, in which one removes a perfect matching from K_n in order to make all degrees even.

Theorem 1 *(a) For every odd $n \geq 3$ and every integer m such that $3 \leq m \leq n$ and*

$$m \mid \frac{n(n-1)}{2},$$

the complete graph K_n is C_m-decomposable.

(b) For every even $n \geq 4$ and every integer m such that $3 \leq m \leq n$ and

$$m \mid \frac{n(n-2)}{2},$$

the graph K_n' is C_m-decomposable, where K_n' denotes the complete graph on n vertices with a perfect matching removed.

The proof of Theorem 1 opened the door to further progress. In 2014, Bryant, Horsley, and Pettersson (Bryant, Horsley, and Pettersson 2014) generalized Šajna's result to decompositions into cycles of *unequal* lengths. They showed that for any odd n and any integers $m_1, \ldots, m_t$ satisfying

$$3 \leq m_i \leq n \quad \text{and} \quad \sum_{i=1}^{t} m_i = \frac{n(n-1)}{2},$$

the complete graph K_n can be decomposed into cycles of lengths $m_1, \ldots, m_t$. Similarly, if n is even and

$$\sum_{i=1}^{t} m_i = \frac{n(n-2)}{2},$$

then K_n can be decomposed into cycles of these lengths together with a perfect matching. This result resolved a conjecture posed by Alspach in 1981.

More generally, given any graph H, we say that a graph G is *H-decomposable* if its edge set can be partitioned into copies of H. Theorem 1 settles the case where $G = K_n$ and $H = C_m$. A closely related setting arises when H is a 2-*factor*, that is, an n-vertex graph in which every vertex has degree 2. Equivalently, a 2-factor is a disjoint union of cycles. The celebrated *Oberwolfach problem* asks for which odd integers n and which n-vertex 2-factors H the complete graph K_n admits an H-decomposition. In 2021, Glock, Joos, Kim, Kühn, and Osthus (Glock et al. 2021) proved that such a decomposition exists for *every* 2-factor H, provided that n is sufficiently large.

References

Bryant, Darryn, Daniel Horsley, and William Pettersson. 2014. "Cycle Decompositions V: Complete Graphs into Cycles of Arbitrary Lengths." *Proc. Lond. Math. Soc. (3)* 108 (5): 1153–92. https://doi.org/10.1112/plms/pdt051.

Glock, Stefan, Felix Joos, Jaehoon Kim, Daniela Kühn, and Deryk Osthus. 2021. "Resolution of the Oberwolfach Problem." *J. Eur. Math. Soc. (JEMS)* 23 (8): 2511–47. https://doi.org/10.4171/jems/1060.

Šajna, Mateja. 2002. "Cycle Decompositions III. Complete Graphs and Fixed Length Cycles." *J. Combin. Des.* 10 (1): 27–78. https://doi.org/10.1002/jcd.1027.