# Cycle decompositions of complete graphs

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This post continues my series on favorite theorems of the 21st century. For a complete overview of the categories and list of my previous selections, see this previous blog post. In the category *Between the Centuries* within Combinatorics, my personal favorite is the cycle decomposition theorem of Šajna.

One of the oldest results in combinatorics is Kirkman's theorem from 1847, which states that for every integer

$$n \equiv 1 \text{ or } 3 \pmod 6,$$

the edges of the complete graph $K_n$ can be partitioned into edge-disjoint triangles. It is straightforward to verify that these congruence conditions on $n$ are necessary. Kirkman's theorem became the starting point of several long-running research programs, some of which spanned more than a century.

One such program asked for decompositions of $K_n$ into cycles $C_m$ of a fixed length $m \geq 3$. We say that a graph $G$ is *$C_m$-decomposable* if its edge set can be written as a disjoint union of cycles of length $m$ (these cycles need not be vertex-disjoint).

A particularly famous special case is $m = n$. A classical result, attributed to Walecki by Lucas in 1892, asserts that for every odd $n$ the complete graph $K_n$ can be decomposed into cycles of length $n$. Such cycles are called Hamiltonian cycles.

There are several obvious necessary conditions for a $C_m$-decomposition of $K_n$ to exist. First, one must have $3 \leq m \leq n$. Second, the number of edges of $K_n$, namely $n(n-1)/2$, must be divisible by $m$. Finally, since every cycle contributes degree 2 at each of its vertices, all vertex degrees in $K_n$ must be even, which forces $n$ to be odd.

In a landmark paper from 2002, Šajna (Šajna 2002) proved that these obvious necessary conditions are, in fact, sufficient. Her result culminated a long sequence of partial advances and brought a century-old project to a definitive conclusion. She also established an analogous statement for even $n$, in which one removes a perfect matching from $K_n$ in order to make all degrees even.

**Theorem 1** *(a) For every odd $n \geq 3$ and every integer $m$ such that $3 \leq m \leq n$ and*

$$m \mid \frac{n(n-1)}{2},$$

*the complete graph $K_n$ is $C_m$-decomposable.*

*(b) For every even $n \geq 4$ and every integer $m$ such that $3 \leq m \leq n$ and*

$$m \mid \frac{n(n-2)}{2},$$

*the graph $K_n'$ is $C_m$-decomposable, where $K_n'$ denotes the complete graph on $n$ vertices with a perfect matching removed.*

The proof of Theorem 1 opened the door to further progress. In 2014, Bryant, Horsley, and Pettersson (Bryant et al. 2014) generalized Šajna's result to decompositions into cycles of *unequal* lengths. They showed that for any odd $n$ and any integers $m_1, \ldots, m_t$ satisfying

$$3 \leq m_i \leq n \quad \text{and} \quad \sum_{i=1}^{t} m_i = \frac{n(n-1)}{2},$$

the complete graph $K_n$ can be decomposed into cycles of lengths $m_1, \ldots, m_t$. Similarly, if $n$ is even and

$$\sum_{i=1}^{t} m_i = \frac{n(n-2)}{2},$$

then $K_n$ can be decomposed into cycles of these lengths together with a perfect matching. This result resolved a conjecture posed by Alspach in 1981.

More generally, given any graph $H$, we say that a graph $G$ is *$H$-decomposable* if its edge set can be partitioned into copies of $H$. Theorem 1 settles the case where $G = K_n$ and $H = C_m$. A closely related setting arises when $H$ is a *2-factor*, that is, an $n$-vertex graph in which every vertex has degree 2. Equivalently, a 2-factor is a disjoint union of cycles. The celebrated *Oberwolfach problem* asks for which odd integers $n$ and which $n$-vertex 2-factors $H$ the complete graph $K_n$ admits an $H$-decomposition. In 2021, Glock, Joos, Kim, Kühn, and Osthus (Glock et al. 2021) proved that such a decomposition exists for *every* 2-factor $H$, provided that $n$ is sufficiently large.

## References

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