

# Cycle decompositions of complete graphs

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This post continues my series on favorite theorems of the 21st century. For a complete overview of the categories and list of my previous selections, see [this previous blog post](#). In the category *Between the Centuries* within Combinatorics, my personal favorite is the cycle decomposition theorem of Šajna.

One of the oldest results in combinatorics is Kirkman's theorem from 1847, which states that for every integer

$$n \equiv 1 \text{ or } 3 \pmod{6},$$

the edges of the complete graph  $K_n$  can be partitioned into edge-disjoint triangles. It is straightforward to verify that these congruence conditions on  $n$  are necessary. Kirkman's theorem became the starting point of several long-running research programs, some of which spanned more than a century.

One such program asked for decompositions of  $K_n$  into cycles  $C_m$  of a fixed length  $m \geq 3$ . We say that a graph  $G$  is  $C_m$ -decomposable if its edge set can be written as a disjoint union of cycles of length  $m$  (these cycles need not be vertex-disjoint).

A particularly famous special case is  $m = n$ . A classical result, attributed to Walecki by Lucas in 1892, asserts that for every odd  $n$  the complete graph  $K_n$  can be decomposed into cycles of length  $n$ . Such cycles are called Hamiltonian cycles.

There are several obvious necessary conditions for a  $C_m$ -decomposition of  $K_n$  to exist. First, one must have  $3 \leq m \leq n$ . Second, the number of edges of  $K_n$ , namely  $n(n-1)/2$ , must be divisible by  $m$ . Finally, since every cycle contributes degree 2 at each of its vertices, all vertex degrees in  $K_n$  must be even, which forces  $n$  to be odd.

In a landmark paper from 2002, Šajna (Šajna 2002) proved that these obvious necessary conditions are, in fact, sufficient. Her result culminated a long sequence of partial advances and brought a century-old project to a definitive conclusion. She also established an analogous statement for even  $n$ , in which one removes a perfect matching from  $K_n$  in order to make all degrees even.

**Theorem 1** (a) For every odd  $n \geq 3$  and every integer  $m$  such that  $3 \leq m \leq n$  and

$$m \mid \frac{n(n-1)}{2},$$

the complete graph  $K_n$  is  $C_m$ -decomposable.

(b) For every even  $n \geq 4$  and every integer  $m$  such that  $3 \leq m \leq n$  and

$$m \mid \frac{n(n-2)}{2},$$

the graph  $K'_n$  is  $C_m$ -decomposable, where  $K'_n$  denotes the complete graph on  $n$  vertices with a perfect matching removed.

The proof of Theorem 1 opened the door to further progress. In 2014, Bryant, Horsley, and Pettersson (Bryant et al. 2014) generalized Šajna’s result to decompositions into cycles of *unequal* lengths. They showed that for any odd  $n$  and any integers  $m_1, \dots, m_t$  satisfying

$$3 \leq m_i \leq n \quad \text{and} \quad \sum_{i=1}^t m_i = \frac{n(n-1)}{2},$$

the complete graph  $K_n$  can be decomposed into cycles of lengths  $m_1, \dots, m_t$ . Similarly, if  $n$  is even and

$$\sum_{i=1}^t m_i = \frac{n(n-2)}{2},$$

then  $K_n$  can be decomposed into cycles of these lengths together with a perfect matching. This result resolved a conjecture posed by Alspach in 1981.

More generally, given any graph  $H$ , we say that a graph  $G$  is  $H$ -decomposable if its edge set can be partitioned into copies of  $H$ . Theorem 1 settles the case where  $G = K_n$  and  $H = C_m$ . A closely related setting arises when  $H$  is a 2-factor, that is, an  $n$ -vertex graph in which every vertex has degree 2. Equivalently, a 2-factor is a disjoint union of cycles. The celebrated *Oberwolfach problem* asks for which odd integers  $n$  and which  $n$ -vertex 2-factors  $H$  the complete graph  $K_n$  admits an  $H$ -decomposition. In 2021, Glock, Joos, Kim, Kühn, and Osthus (Glock et al. 2021) proved that such a decomposition exists for every 2-factor  $H$ , provided that  $n$  is sufficiently large.

## References

- Bryant, Darryn, Daniel Horsley, and William Pettersson. 2014. “Cycle Decompositions V: Complete Graphs into Cycles of Arbitrary Lengths.” *Proc. Lond. Math. Soc.* (3) 108 (5): 1153–92. <https://doi.org/10.1112/plms/pdt051>.

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- Šajna, Mateja. 2002. "Cycle Decompositions III. Complete Graphs and Fixed Length Cycles." *J. Combin. Des.* 10 (1): 27–78. <https://doi.org/10.1002/jcd.1027>.