Diophantine approximation on fractal sets

Bogdan Grechuk • 8 Jan 2026

An exciting recent development in Diophantine approximation is the paper (Bénard, He, and Zhang 2026), which has just been accepted to the *Journal of the American Mathematical Society* (JAMS). In this work, Bénard, He, and Zhang make a major advance in our understanding of how well irrational numbers can be approximated by rational ones, even if one restricts attention to irrational numbers from fractal sets such as the Cantor set.

One of the earliest theorems in mathematics asserts that the diagonal of a square of side length 1 cannot be measured exactly using any unit obtained by subdividing the centimetre into equal parts. In modern language, this is the statement that $\sqrt{2}$ is irrational. Since then, a central question in number theory has been to quantify how well an irrational number α can be approximated by rational numbers $\frac{m}{n}$. We may (and will) always assume that $n > 0$.

There is an inherent trade-off between the quality of the approximation and the size of the denominator n. A classical result from the theory of continued fractions, which is also a corollary of Dirichlet's famous *approximation theorem*, states that for any irrational $\alpha \in \mathbb{R}$ the inequality

$$0 < \left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2} \tag{1}$$

is satisfied for infinitely many pairs of integers m and n.

In 1891, Hurwitz sharpened this result. He proved that for $k = \sqrt{5}$ and every $\alpha \in \mathbb{R}$ there exist infinitely many integers m, n such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{kn^2}. \tag{2}$$

Hurwitz's theorem is optimal: for the golden ratio

$$\alpha = \frac{1+\sqrt{5}}{2},$$

and for any $k > \sqrt{5}$, inequality (2) has only finitely many solutions in integers m, n.

A natural refinement of this question is to ask what level of approximation is achieved for a "typical" real number, rather than for all real numbers. Let $\psi : \mathbb{N} \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$. We say that a real number α is ψ-*approximable* if the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n} \tag{3}$$

has infinitely many solutions in integers m, n. Denote by $S(\psi)$ the set of all ψ-approximable real numbers.

In a landmark 1924 paper, Khintchine (Khintchine 1924) proved the following fundamental theorem.

1. If $\sum_{n=1}^{\infty} \psi(n) = \infty$ and the sequence $n\psi(n)$ is non-increasing, then Lebesgue-almost every $\alpha \in \mathbb{R}$ belongs to $S(\psi)$.
2. Conversely, if $\sum_{n=1}^{\infty} \psi(n) < \infty$, then Lebesgue-almost every $\alpha \in \mathbb{R}$ does not belong to $S(\psi)$.

For instance, applying part (i) with $\psi(n) = \frac{1}{kn}$ shows that for any $k > 0$ and for almost all $\alpha \in \mathbb{R}$, inequality (2) has infinitely many solutions in integers m, n.

In 1984, Mahler (Mahler 1984) posed a striking question: how well can irrational points in Cantor's middle-third set be approximated by rational numbers? Recall that the Cantor set $C_3 \subset [0, 1]$ consists of those real numbers whose base-3 expansions omit the digit 1. Since C_3 has Lebesgue measure zero, Khintchine's original theorem does not apply in this setting.

The recent paper (Bénard, He, and Zhang 2026) not only resolves Mahler's question completely, but does so in a far more general framework.

Recall that the *Borel σ-algebra* $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all open subsets of $\mathbb{R}$. A *Borel probability measure* σ on $\mathbb{R}$ is a measure defined on $\mathcal{B}(\mathbb{R})$ with total mass $\sigma(\mathbb{R}) = 1$. Such a measure is called *self-similar* if it satisfies

$$\sigma(A) = \sum_{i=1}^{m} \lambda_i \, \sigma\big(\varphi_i^{-1}(A)\big) \quad \text{for all Borel sets } A \subset \mathbb{R}, \tag{4}$$

where $m \geq 1$, the weights $\lambda_1, \ldots, \lambda_m$ are positive and sum to 1, and the maps $\varphi_1, \ldots, \varphi_m : \mathbb{R} \to \mathbb{R}$ are invertible affine contractions with no common fixed point.

Theorem 1 *Let σ be a self-similar probability measure on $\mathbb{R}$, and let $\psi : \mathbb{N} \to (0, \infty)$ be a non-increasing function. Then*

$$\sigma\big(S(\psi)\big) = \begin{cases} 0, & \textit{if } \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1, & \textit{if } \sum_{n=1}^{\infty} \psi(n) = \infty. \end{cases}$$

As an immediate consequence, consider the self-similar measure supported on the Cantor set C_3, obtained by taking $m = 2$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, and

$$\varphi_1(t) = \frac{t}{3}, \qquad \varphi_2(t) = \frac{t+2}{3}.$$

Theorem 1 then implies that almost every $\alpha \in C_3$ is ψ-approximable if $\sum_{n=1}^{\infty} \psi(n) = \infty$, and that almost no $\alpha \in C_3$ is ψ-approximable if the series converges. This provides a complete and definitive answer to Mahler's question.

References

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