The density version of Hindman’s finite sums theorem

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The first issue of *Inventiones Mathematicae* for 2026 features a paper by Kra, Moreira, Richter, and Robertson (Kra et al. 2026) that resolves a long-standing problem in additive combinatorics: they establish the strongest possible *density version* of Hindman’s finite sums theorem.

To place this result in context, recall Schur’s classical theorem from 1916. It asserts that for any finite colouring of the integers, there exist integers x, y, z of the same colour satisfying

$$x + y = z.$$

In 1974, Hindman proved a far-reaching generalization of this phenomenon. His theorem states that for any finite colouring of the integers, there exists an infinite set A such that *all* finite sums of distinct elements of A,

$$S_A = \left\{ \sum\nolimits_{x \in B} x \,\middle|\, B \subset A,\ |B| < \infty \right\},$$

receive the same colour.

A notable corollary of Hindman’s theorem is that for any finite colouring of the positive integers, one can find infinite sets $B, C \subset \mathbb{N}$ such that the sumset

$$B + C = \{b + c :\ b \in B,\ c \in C\}$$

is monochromatic. Motivated by this, Erdős conjectured that an analogous statement should hold in a density setting: every subset $A \subset \mathbb{N}$ of positive upper density should contain a sumset of the form $B + C$ with B and C infinite. This conjecture, now known as the *Erdős sumset conjecture*, was confirmed in 2019 by Moreira, Richter, and Robertson (Moreira, Richter, and Robertson 2019).

Subsequent work strengthened this result. In 2024, Kra, Moreira, Richter, and Robertson (Kra et al. 2024) proved that for any set $A \subset \mathbb{N}$ of positive upper density and any integer $k \geq 1$, there exist infinite sets

$$B_1, \ldots, B_k \subset \mathbb{N}$$

such that

$$B_1 + \cdots + B_k = \{b_1 + \cdots + b_k :\ b_i \in B_i\} \subset A.$$

More recently, a 2025 preprint by Hernández, Kousek, and Radić (Hernández, Kousek, and Radić 2025) extended this further: for any set $A \subset \mathbb{N}$ of positive upper density, there exists an infinite sequence of infinite sets $B_1, B_2, \ldots$ such that

$$B_1 + \cdots + B_k \subset A \quad \text{for all } k \geq 1.$$

All of the results above can be viewed as density versions of consequences of Hindman's original theorem. As early as 1975, Erdős asked whether one could formulate and prove a density analogue of the *full* Hindman theorem itself. The recent paper (Kra et al. 2026) answers this question in the strongest possible form.

Theorem 1 *Let $C \subset \mathbb{N}$ be a set of positive upper density, and let $k \geq 1$ be an integer. Then there exist an infinite set $A \subset \mathbb{N}$ and an integer $t \geq 0$ such that*

$$S_{A,k} = \left\{ \sum\nolimits_{x \in B} x \,\middle|\, B \subset A,\ |B| \leq k \right\} \subset C - t.$$

In comparison with Hindman's theorem, Theorem 1 involves two essential restrictions. First, only sums of at most k elements are considered. Second, these sums are required to lie in a *translate* of C, rather than in C itself. Both limitations are known to be unavoidable. For instance, the set of odd integers shows that a shift t is necessary, while Straus constructed examples of sets C with density arbitrarily close to 1 that do not contain any translate of S_A for an infinite set A.

References

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