

# Boaz Klartag's talk on Sphere packing

Bogdan Grechuk • 6 Jan 2026

I am pleased to invite you to the first talk in a new series of online seminars devoted to accessible presentations of some of the most significant mathematical theorems of the 21st century.

The inaugural talk will take place on **Wednesday, January 28, 2026, at 4:00 PM (UK time)**. **Prof. Boaz Klartag** (Weizmann Institute of Science, Israel) will speak on the topic of *sphere packing*.

To join the talk, just [click here](#) at the scheduled time. No registration required. For a full list of upcoming seminars, please visit the seminar webpage: <https://th21.le.ac.uk/next-talks/>.

For the announcements of other talks, as well as descriptions of other recent great math theorems, please see [the full list of my posts in this blog](#).

## The Sphere Packing Problem.

What is the densest possible way to pack congruent balls in  $\mathbb{R}^d$ ? More precisely, for a center  $A \in \mathbb{R}^d$  and radius  $R > 0$ , an *open ball*  $B(A, R)$  is defined by

$$B(A, R) = \{X \in \mathbb{R}^d : |AX| < R\},$$

where  $|\cdot|$  denotes the Euclidean distance. A *packing of congruent balls*, or *sphere packing*, is an (infinite) collection  $E$  of non-overlapping open balls of equal radius.

Fix a point  $O \in \mathbb{R}^d$ . The *upper density* of a packing  $E$  is defined as

$$\delta(E) = \limsup_{R \rightarrow \infty} \frac{|B(O, R) \cap E|}{|B(O, R)|}, \quad (1)$$

where  $|\cdot|$  denotes  $d$ -dimensional volume. It can be shown that  $\delta(E)$  depends only on  $E$  and not on the choice of  $O$ . If the limit in (1) exists, then  $\delta(E)$  is simply called the *density* of  $E$ .

In  $\mathbb{R}^2$ , a natural arrangement of non-overlapping circles is given by the *regular hexagonal packing*, which has density  $\frac{\pi}{\sqrt{12}} = 0.9069\dots$ . In 1773, Lagrange proved that this packing is optimal among *lattice packings*, namely those packings whose centers form a lattice. Recall that a *lattice*  $\Lambda \subset \mathbb{R}^d$  is a set of the form

$$\Lambda = \left\{ \sum_{i=1}^d a_i v_i \mid a_i \in \mathbb{Z} \right\},$$

where  $v_1, \dots, v_d$  is a basis of  $\mathbb{R}^d$ . In 1910, Thue gave a (partly non-rigorous) argument showing that the regular hexagonal packing is optimal among all packings in  $\mathbb{R}^2$ . A fully rigorous proof was later provided by Tóth in 1943.

In  $\mathbb{R}^3$ , two classical arrangements—the *cubic close packing* and the *hexagonal close packing*—achieve the same density  $\frac{\pi}{3\sqrt{2}}$ . In 1611, Kepler conjectured that these are the densest possible packings. This conjecture was finally confirmed by Hales in 2005 (Hales 2005).

The sphere packing problem has been completely solved only in a few other dimensions. In 2017, Viazovska (Viazovska 2017) resolved the problem in dimension  $d = 8$ , and shortly thereafter Cohn *et al.* (Cohn et al. 2017) did so in dimension  $d = 24$ . For dimensions  $d \neq 1, 2, 3, 8, 24$ , the determination of the optimal packing density  $\rho_d^*$  remains open.

In 2022, Cohn, de Laat, and Salmon (Cohn et al. 2022) developed new methods improving the previously best known upper bounds on  $\rho_d^*$  in dimensions  $4 \leq d \leq 7$  and  $9 \leq d \leq 16$ .

For general  $d$ , Kabatyansky and Levenshtein showed in 1978 (Kabatyanskiĭ and Levenshtein 1978) that

$$\rho_d^* \leq 2^{(-0.599\dots+o(1))d}, \quad (2)$$

a bound that remains essentially optimal to this day, up to constant-factor improvements.

By contrast, the best known lower bounds are much weaker. In 1947, Rogers (Rogers 1947) proved that

$$\rho_d^* \geq c d 2^{-d}$$

for some universal constant  $c > 0$ . This bound stood for more than 75 years, apart from improvements to the constant. In 2023, Campos, Jenssen, Michelen, and Sahasrabudhe (Campos et al. 2023) improved Rogers' bound by a factor of  $\log d$ . In 2026, Klartag (Klartag 2026) developed a construction that beats Rogers' construction by a factor of  $d$ .

**Theorem 1** *There exists a constant  $c > 0$  such that, for all  $d \geq 1$ ,*

$$\rho_d^* \geq c d^2 2^{-d}.$$

Remarkably, the construction underlying Theorem 1 is a *lattice packing*. Klartag (Klartag 2026) further speculates that, at least within the class of lattice packings, the lower bound in Theorem 1 may be essentially optimal, up to a logarithmic correction.

**Update 08.06.2026.** Paper (Klartag 2026) has now been published. The reference has been updated to the published version. Also, Abuya, Gargava and Zhao (Abuya et al. 2026) proved that  $\rho_d^* \geq cd^2 \log \log d \cdot 2^{-d}$  for infinitely many values of  $d$ , and their example is again a lattice packing.

## References

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