Boaz Klartag’s talk on Sphere packing

Bogdan Grechuk • 6 Jan 2026

I am pleased to invite you to the first talk in a new series of online seminars devoted to accessible presentations of some of the most significant mathematical theorems of the 21st century.

The inaugural talk will take place on **Wednesday, January 28, 2026, at 4:00 PM (UK time)**. **Prof. Boaz Klartag** (Weizmann Institute of Science, Israel) will speak on the topic of *sphere packing*.

To join the talk, just click here at the scheduled time. No registration required. For a full list of upcoming seminars, please visit the seminar webpage: https://th21.le.ac.uk/next-talks/.

For the announcements of other talks, as well as descriptions of other recent great math theorems, please see the full list of my posts in this blog.

The Sphere Packing Problem.

What is the densest possible way to pack congruent balls in $\mathbb{R}^d$? More precisely, for a center $A \in \mathbb{R}^d$ and radius $R > 0$, an *open ball* $B(A, R)$ is defined by

$$B(A,R) = \{X \in \mathbb{R}^d : |AX| < R\},$$

where $|\cdot|$ denotes the Euclidean distance. A *packing of congruent balls*, or *sphere packing*, is an (infinite) collection E of non-overlapping open balls of equal radius.

Fix a point $O \in \mathbb{R}^d$. The *upper density* of a packing E is defined as

$$\delta(E) = \limsup_{R\to\infty} \frac{|B(O,R) \cap E|}{|B(O,R)|}, \tag{1}$$

where $|\cdot|$ denotes d-dimensional volume. It can be shown that $\delta(E)$ depends only on E and not on the choice of O. If the limit in (1) exists, then $\delta(E)$ is simply called the *density* of E.

In $\mathbb{R}^2$, a natural arrangement of non-overlapping circles is given by the *regular hexagonal packing*, which has density $\frac{\pi}{\sqrt{12}} = 0.9069\dots$. In 1773, Lagrange proved that this packing is optimal among *lattice packings*, namely those packings whose centers form a lattice. Recall that a *lattice* $\Lambda \subset \mathbb{R}^d$ is a set of the form

$$\Lambda = \left\{ \sum_{i=1}^{d} a_i v_i \;\middle|\; a_i \in \mathbb{Z} \right\},$$

where $v_1, \ldots, v_d$ is a basis of $\mathbb{R}^d$. In 1910, Thue gave a (partly non-rigorous) argument showing that the regular hexagonal packing is optimal among all packings in $\mathbb{R}^2$. A fully rigorous proof was later provided by Tóth in 1943.

In $\mathbb{R}^3$, two classical arrangements—the *cubic close packing* and the *hexagonal close packing*—achieve the same density $\frac{\pi}{3\sqrt{2}}$. In 1611, Kepler conjectured that these are the densest possible packings. This conjecture was finally confirmed by Hales in 2005 (Hales 2005).

The sphere packing problem has been completely solved only in a few other dimensions. In 2017, Viazovska (Viazovska 2017) resolved the problem in dimension $d = 8$, and shortly thereafter Cohn *et al.* (Cohn et al. 2017) did so in dimension $d = 24$. For dimensions $d \neq 1, 2, 3, 8, 24$, the determination of the optimal packing density ρ_d^* remains open.

In 2022, Cohn, de Laat, and Salmon (Cohn, Laat, and Salmon 2022) developed new methods improving the previously best known upper bounds on ρ_d^* in dimensions $4 \leq d \leq 7$ and $9 \leq d \leq 16$.

For general d, Kabatyansky and Levenshteın showed in 1978 (Kabatyanskiı and Levenshteın 1978) that

$$\rho_d^* \leq 2^{(-0.599\ldots+o(1))d}, \tag{2}$$

a bound that remains essentially optimal to this day, up to constant-factor improvements.

By contrast, the best known lower bounds are much weaker. In 1947, Rogers (Rogers 1947) proved that

$$\rho_d^* \geq c\, d\, 2^{-d}$$

for some universal constant $c > 0$. This bound stood for more than 75 years, apart from improvements to the constant. In 2023, Campos, Jenssen, Michelen, and Sahasrabudhe (Campos et al. 2023) improved Rogers' bound by a factor of $\log d$. More recently, in 2025, Klartag (Klartag 2025) developed a construction that beats Rogers' construction by a factor of d.

Theorem 1 *There exists a constant $c > 0$ such that, for all $d \geq 1$,*

$$\rho_d^* \geq c\, d^2\, 2^{-d}.$$

Remarkably, the construction underlying Theorem 1 is a *lattice packing*. Klartag (Klartag 2025) further speculates that, at least within the class of lattice packings, the lower bound in Theorem 1 may be essentially optimal, up to the value of the universal constant c or possibly a logarithmic correction.

References

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